Needham, Chapter 3, Exercise 17, p. 185-186.
G. Palmer, August 8, 2021, 9:50 am PDT

Out[414]=

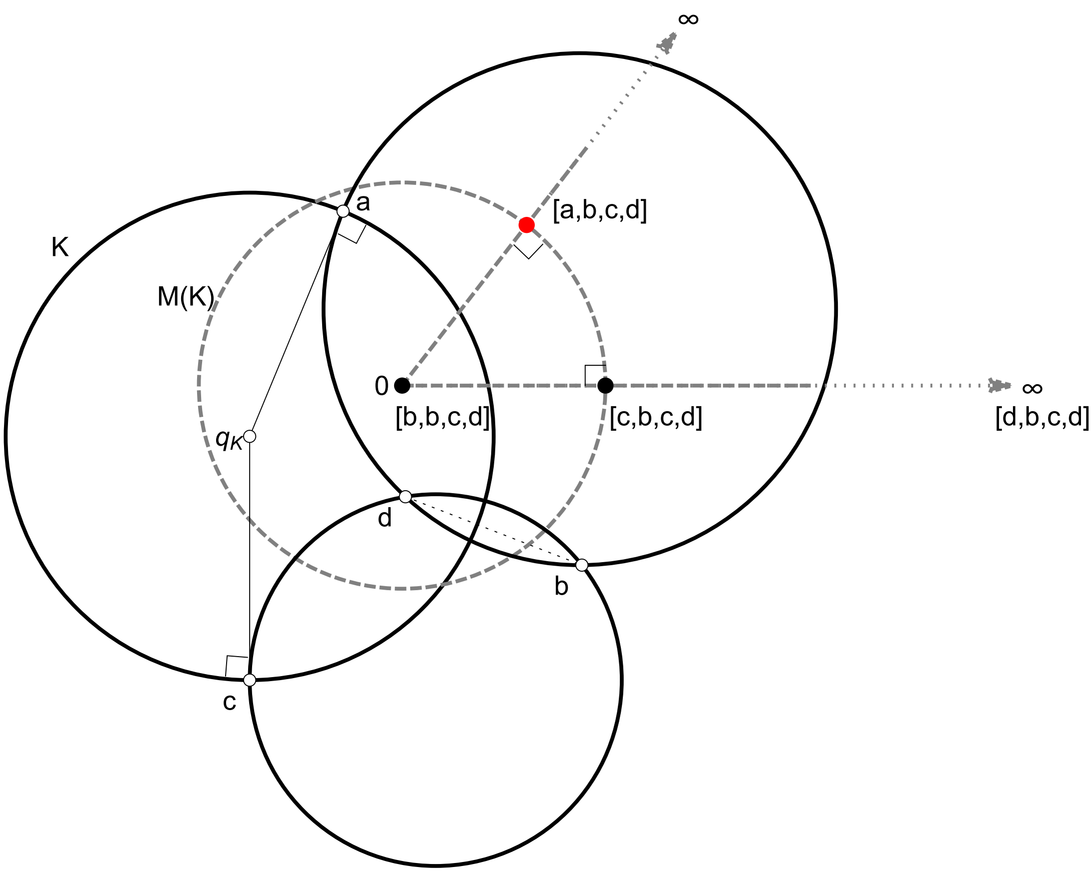


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Figure 1. Möbius transformation [a,b,c,d]
of orthogonal circles K, cbd, and abd
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*17 Show geometrically that if a and c lie on a circle K, and b and d are symmetric with respect to K, then the point [a, b, c, d] lies on the unit circle. [Hint: Draw the two circles through a, b, d and b, c, d. Now think of [z, b, c, d] as a Möbius transformation.*

To solve this problem, one should apply the most general framework in complex analysis: The transformation M(z) = [z, b, c, d] applies not just to this circle or that circle, but to the complex plane, so we can conveniently think of the pre-image and image, the z-plane and w-plane (here instantiated as the M(z) plane). We can say that M "maps" pre-image to image. In Figure 1, circles and straight lines in the image plane are plotted with thick, gray, dashed lines, continued with dotted lines. Solid lines are in the pre-image.

The exercise introduces three circles, two of which contain the inversion points b and d, so those two circles are orthogonal to K, the circle of inversion. Circle K (which contains neither b nor d), maps to a new circle M(K). By the construction of the cross-ratio, the two orthogonal circles map to two infinite lines on which M maps b and d to 0 and ∞ respectively. Points c and a occur in different orthogonal image circles, so they map to points on separate infinite lines. Nevertheless, both c and a map into a line passing through 0 and ∞ and both map into M(K). By the cross-ratio, c maps to 1 in the real line from 0 to ∞. Because Möbius transformations are conformal, the angles between the infinite lines and M(K) must preserve the angles between the orthogonal circles and K. Hence, the angles at M(c) and M(a) between M(K) and the two infinite lines must both be right angles. Since both lines pass through 0, a must fall on the unit circle.

Out[552]=

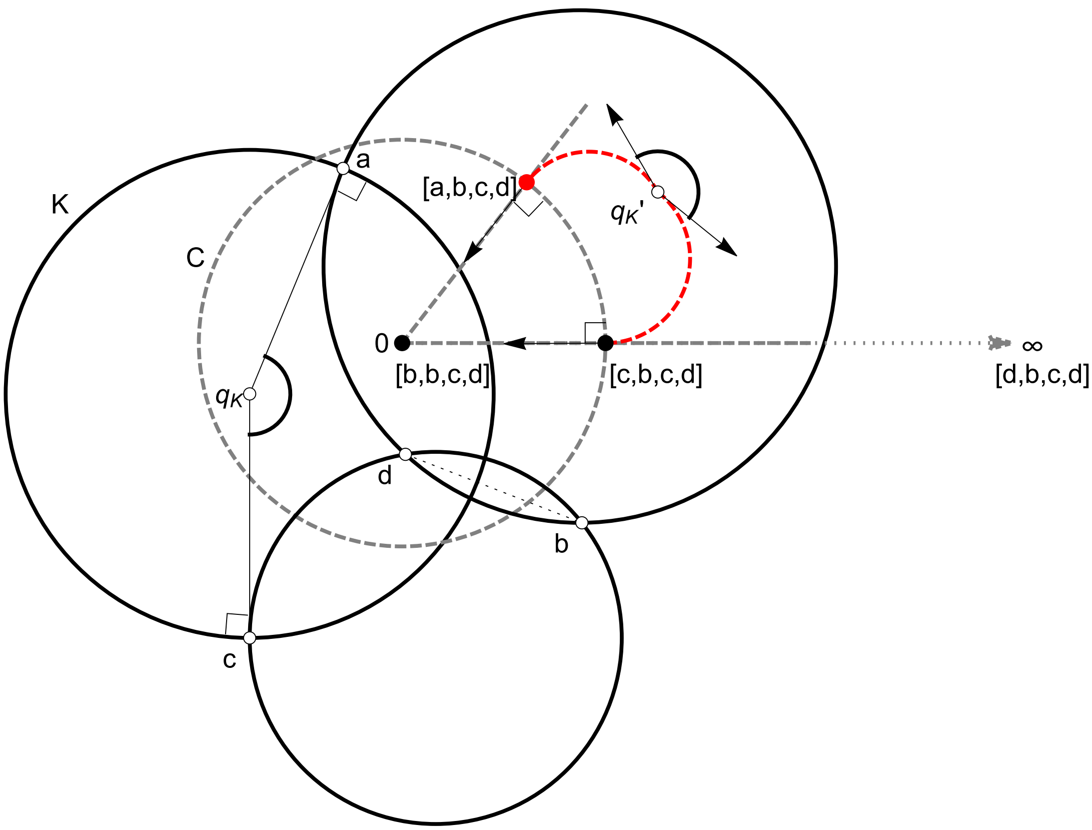


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Figure 2. Preservation of angles
Möbius transformation [a,b,c,d]
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Figure 1 makes it appear that angles are preserved at M(a) = [a,b,c,d] and M(c) = [c,b,c,d], but not at 0. In fact angles between rays at $M(q_K)$ should not be preserved at 0, because 0 is not $q_K' = M(q_K) = [q_K, b, c, d]$. By plotting $q_K'$ and plotting the transformed rays (a – $q_K$) and (c – $q_K$) as curves (red), we can see that angles are preserved at both the the transformed centre $q_K'$ = M(K) and at the intersections of the curves with the unit circle at M(a) and M(c). Since M is conformal, the orientations are also preserved.

To program the curves, the rays were parameterized and M(z) was applied. The angles $\angle(a - q_K - c)$ and $\angle(M(a) - q_K' - M(c))$ were estimated by finding the direction of the second point of M(z) applied to each of the parameterized rays.

$$\angle(a - q_k - c) \approx 2.78$$
$$\angle(M(a) - q_K{}' - M(c)) \approx 2.75$$

Given the crude method of estimation, this result supports the preservation of angles. The tangents at M(a) and M(c) were estimated by finding the direction of the last points of M(z) applied to the rays.

It also follows from this construction that if b and d are symmetrical in a circle K, and c is a point at the intersection of K and $C_{\mathrm{bcd}}$, the cross-ratio [z, b, c, d] maps K to the unit circle.