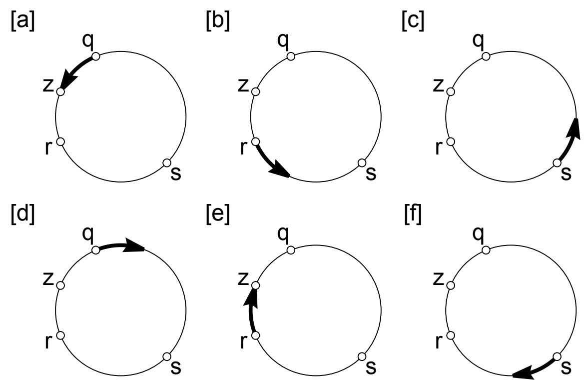
Needham, Chapter 3, Exercise 16, p. 185-186.
G. Palmer, July 18, 2021, 10:30 pm PDT

Out[∘]=



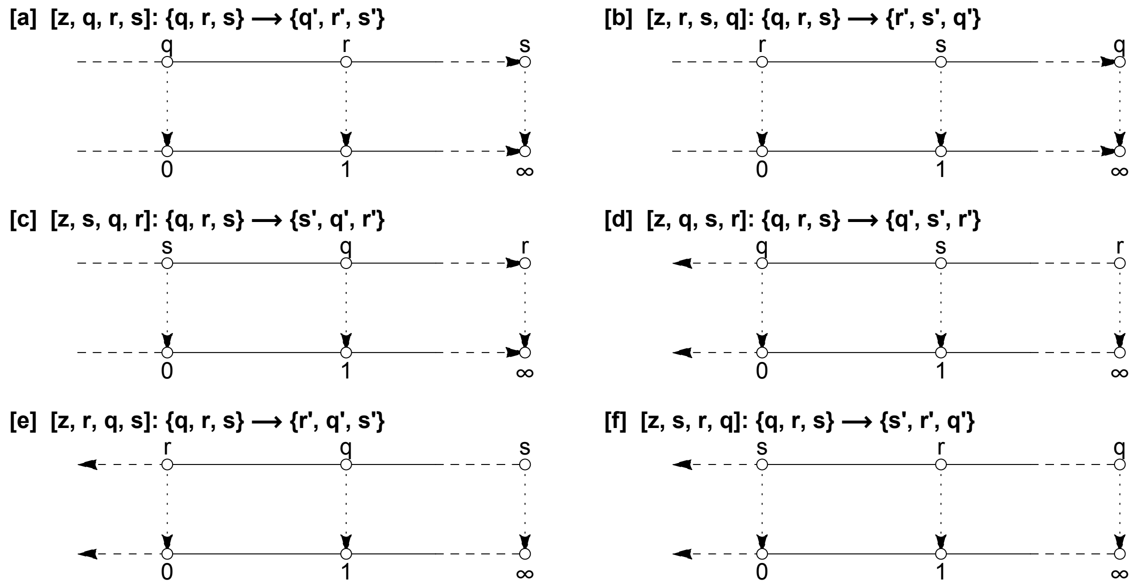
`Figure 1. Permutations of q, r, s as starting points in [z, q, r, s] on oriented circle`

16. As in figure [28], think of the cross-ratio as a Möbius transformation.

(i) Explain geometrically why permuting q, r, s, in [z, q, r, s] yields six different transformations.

Figure 1 provides a way of considering the geometry of the permutations based on (30) and the left hand figure, p. 155, which shows the unique circle through the points q, r, s, *oriented so that these points succeed each other in the stated order*. This corresponds to the Möbius transformation in (29). The insistence on orientation and order suggests that we can rearrange the order to obtain three Möbius transformations [a-c] and we can reverse the orientation to obtain three more [d-f]. In figure 2, the directions of the thick arrows indicate orientations of the circles. If we take the starting point for the sequence of qrs to be the starting point of the arrow in each of [a-f] in Figure 1, then we have the six transformations [z, q, r, s], [z, r, s, q], [z, s, q, r], [z, q, s, r], [z, r, q, s], and [z, s, r, q].

Out[•]=



`Figure 2. Mapping q, r, s by permutations of {q, r, s}`
`in [z, q, r, s] to permutations of q', r', s' = 0, 1, ∞`

Perhaps a better way to conceptualize the Möbius transformations is presented in Figure 2, showing six permutations of q, r, s in [z, q, r, s] corresponding to the six circles in Figure 1. The top arrow in each plot can be thought of as a segment of a finite or infinite circle opened up and flattened. Orientation of the circles is shown by the horizontal arrows. If we let q’, r’, s’ = 0, 1, ∞, the bottom line is the real line and the mappings are from the top line to the bottom line as shown by the vertical dashed arrows. The arrow on the bottom line can be thought of as a segment of an infinite circle with either positive (pointing right) or negative (pointing left) orientation. For each permutation of q, r, s in [z, q, r, s], there is a mapping of q, r, or s (used individually as arguments for z) to 0, 1, ∞ as shown for each in figures [a] to [f]. This mapping of q, r or s to q’, r’, s’ = 0, 1, ∞ reveals orientation of the mapping to be the same as that of the permutation. For example, one can say that [z, s, r, q] has negative orientation (Figures 1[f] and 2[f]), and its mapping from {0, 1, ∞} ⟶ {∞, 1, 0} also has negative orientation. One must think of the bottom arrow as traversing the infinite circle from ∞ to 1 to 0 and back towards ∞.

(ii) If I(z) is the Möbius transformation that leaves 1 fixed and that swaps 0 with ∞, explain geometrically why I ○ [z, q, r, s] = [z, s, r, q].

[z, s, r, q], seen in the last diagram in Figure 2, reverses the orientation and begins at s. The exercise seems to assume that {q’, r’, s’} = {0, 1, ∞} [Curley brackets indicate an ordered list.] [z, q, r, s] is the *Möbius transformation* that maps each of the elements 0, 1, ∞ to itself.* The mapping provides the arguments to z for three mappings, which can be seen in the last diagram in Figure 2. Let q = 0, r = 1, and s = ∞:

[q, s, r, q] = [0, s, r, q] = s’ = ∞

[r, s, r, q] = [1, s, r, q] = r' = 1
[s, s, r, q] = [∞, s, r, q]= q' = 0

Then [z, s, r, q] ∘ {q, r, s} = {s', r', q'} = {∞, 1, 0}. Thus, as discussed in part (i), [z, s, r, q] provides the desired transformation 0, 1, ∞ ⟶ ∞, 1, 0. *Since [z, s, q, r] is an analytic function, we can extend it from the real line to all of $\mathbb{C}$.*

Geometrically, the reverse permutation from qrs to srq results in complex inversion in the unit circle in $\mathbb{C}$. That is, [z, s, r, q] = 1/z.

(iii) If J(z) is the Möbius transformation that sends 0, 1, ∞ to 1, ∞, 0, respectively, explain geometrically why J ○ [z, q, r, s] = [z, s, q, r].

Again, when applied to q, r, s, [z, q, r, s] is an identity mapping. So J ○ [z, q, r, s]({q, r, s}) is equivalent to J({q, r, s}), where this notation suggests that J is applied successively to each of q, r, s in that order. [z, s, q, r] is illustrated in the third diagram in Figure 2. This permutation lends the circle positive (anticlockwise) orientation, as seen in Figures 1[c] and 2[c], and it provides the specified transformation. Figure 2 shows that [z, s, q, r] maps {0, 1, ∞} to {1, ∞,0}. Since [z, s, q, r] is an analytic function, we can extend it from the real line to all of $\mathbb{C}$.

[0, s, q, r] = q' = 1
[1, s, q, r] = r' = ∞
[∞, s, q, r] = s' = 0

(iv) Employing the abbreviation $\mathcal{X}$= [z, q, r, s], explain why the six Möbius transformations in part (i) can be expressed as

$\mathcal{X}$, I ○ $\mathcal{X}$, J ○ $\mathcal{X}$, I ○ J ○ $\mathcal{X}$, J ○ I ○ $\mathcal{X}$, I ○ J ○ I ○ $\mathcal{X}$

(1) $\mathcal{X}$ ≡ [z, q, r, s]

(2) I ○ $\mathcal{X}$ = I ○ [z, q, r, s] = [z, s, r, q] (from part ii)

(3) J ○ $\mathcal{X}$ = J ○ [z, q, r, s] = [z, s, q, r] (from part iii)

(4) I ○ J ○ $\mathcal{X}$ = I ○ [z, s, q, r] = [z, r, q, s] (by swapping first and last in (3))

(5) J ○ I ○ $\mathcal{X}$ = J ○ [z, s, r, q] = [z, q, s, r] (by moving last to first in (2))

(6) I ○ J ○ I ○ $\mathcal{X}$ = I ○ [z, q, s, r] = [z, r, s, q] (by swapping first and last in (5))

These are the same as the transformations in (i).

(1) (6) (3) (5) (4) (2)
[z, q, r, s], [z, r, s q], [z, s q, r], [z, q, s, r], [z, r, q, s], [z, s, r, q]

(v) Show that I(z) = (1/z) and J(z) = 1/(1–z).

Substitute q = 0, r = 1, s = ∞.

$$I(z) = [z, s, r, q] = \frac{(z-s)\,(r-q)}{(z-q)\,(r-s)} = \frac{(z-\infty)}{z(1-\infty)} = \frac{-\infty}{z-z\infty} \equiv \frac{-\infty}{-z\infty} = \frac{1}{z}$$

$$J(z) = [z, s, q, r] = \frac{(z-s)\,(q-r)}{(z-r)\,(q-s)} = \frac{(z-\infty)\,(0-1)}{(z-1)\,(0-\infty)} = \frac{-(z-\infty)}{-\infty(z-1)} \equiv \frac{\infty}{-\infty(z-1)} = \frac{-1}{(z-1)} = \frac{1}{(1-z)}$$

It appears that explicit calculation of [0, s , r , q] where q = 0, r = 1, s = ∞ results in –∞ rather than +∞. But using [z, s, r, q] = 1/z, the result is [0, s, r, q] = 1/0 = ∞ as desired. I don't know how to resolve the contradiction except to suggest the -∞ means the same thing as ∞.

(vi) Deduce that the six possible values of the cross-ratio are

$$\mathcal{X},\quad \frac{1}{\mathcal{X}},\quad \frac{1}{1-\mathcal{X}},\quad 1-\mathcal{X},\quad \frac{\mathcal{X}}{\mathcal{X}-1},\quad \frac{\mathcal{X}-1}{\mathcal{X}}$$

$\mathcal{X} \equiv$ [z, q, r, s], by definition

To obtain the remaining five, substitute the results of part (iv) into I(z) and J(z).

$$I \circ \mathcal{X} = \frac{1}{\mathcal{X}}$$

$$J \circ \mathcal{X} = \frac{1}{(1-\mathcal{X})}$$

$$I \circ J \circ \mathcal{X} = 1/(J \circ \mathcal{X}) = 1\text{-}\mathcal{X}$$

$$J \circ I \circ \mathcal{X} = \frac{1}{(1-I \circ \mathcal{X})} = \frac{1}{(1-1/\mathcal{X})} = \frac{\mathcal{X}}{\mathcal{X}-1}$$

$$I \circ J \circ I \circ \mathcal{X} = 1/\left(\frac{\mathcal{X}}{(\mathcal{X}-1)}\right) = \frac{\mathcal{X}-1}{\mathcal{X}}$$

Footnotes

*Where {q’, r’, s’} = {0, 1, ∞}, it would make more sense to me to write *[z, q’, r’, s’] is the Möbius transformation that maps each of the elements 0, 1, ∞ to itself.*