Needham, Chapter 3, Exercise 15, p. 185.
G. Palmer, July 15, 2021, 12 noon PDT

Out[105]=

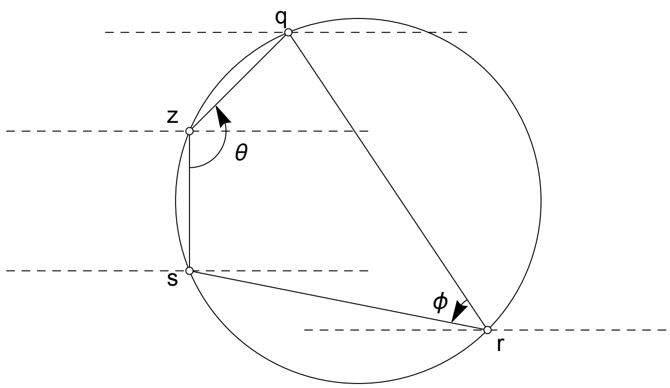

Figure 1. Based on left figure in this exercise.

15 Show that in both of the figures below, Arg[z, q, r, s] = $\theta + \phi$*. Hence, deduce (31), p. 155.*

(31) says *A point p lies on the circle C through q, r, s if and only if*

Im[p, q, r, s] = 0

Im[p, q, r, s] = 0 if and only if Arg[p, q, r, s] is 0 or π.

Equation (1) will be useful for comparison to $\theta + \phi$.

$$Arg[z, q, r, s] = \text{Arg}\left[\frac{(z-q)\,(r-s)}{(z-s)\,(r-q)}\right] \quad (\text{p. } 154)$$

$$= \text{Arg}[z-q] + \text{Arg}[r-s] - \text{Arg}[z-s] - \text{Arg}[r-q] \qquad (1)$$

By inspecting Figure 1, one can write the following equations:

$$\theta = \text{Arg}[q-z] - \text{Arg}[s-z]$$

$$\phi = \text{Arg}[s-z] - \text{Arg}[q-z]$$

$$\theta + \phi = \text{Arg}[q-z] - \text{Arg}[s-z] + \text{Arg}[s-r] - \text{Arg}[q-r]$$

$$= \text{Arg}[q-z] + \text{Arg}[s-r] - \text{Arg}[s-z] - \text{Arg}[q-r]$$

$$= (\text{Arg}[z-q] - \pi) + (\text{Arg}[r-s] + \pi) - (\text{Arg}[z-s] - \pi) - (\text{Arg}[r-q] + \pi) \qquad (2)$$

Note that all the angles in parentheses in (2) fall in the range [$-\pi$ .. π].

= Arg[z–q] + Arg[r–s] – Arg[z–s] – Arg[r–q] = *Arg*[*z*, *q*, *r*, *s*]   cf. (1)

By inspecting the right hand figure in this exercise in Needham, one can write,

θ = Arg[s–z] – Arg[q–z]

ϕ = Arg[s–r] – Arg[q–r]

$\theta + \phi$ = Arg[s–z] – Arg[q–z] + Arg[s–r] – Arg[q–r]

= (Arg[z–s] – π) – (Arg[z–q] – π) + (Arg[r–s] – π) – (Arg[r–q] – π)

= Arg[z–q] + Arg[r–s] – Arg[z–q] – Arg[r–q] = *Arg*[*z*, *q*, *r*, *s*]

We have shown that in both figures, $\theta + \phi$ = *Arg*[*z*, *q*, *r*, *s*].

We have shown that when four points are randomly located on a circle, the sum of the angles θ and ϕ between two chords drawn at any two of the four points equals the argument of the cross–ratio. It follows that the sum of the remaining two angles also equals the argument of the cross ratio. In Figure 1, we can see that sum of all four interior angles must equal 2π. Hence the Arg of the cross–ratio must equal π, so, substituting p for z, Im[p, q, r, s] = 0. From this we can deduce that *a point p lies on the circle C through q, r, s if and only if Im[p, q, r, s] = 0.*