Needham, Ch 3, Ex. 14, pp. 184-185
G. Palmer, June 28, 2021, 5:30 pm PDT

Out[28]=

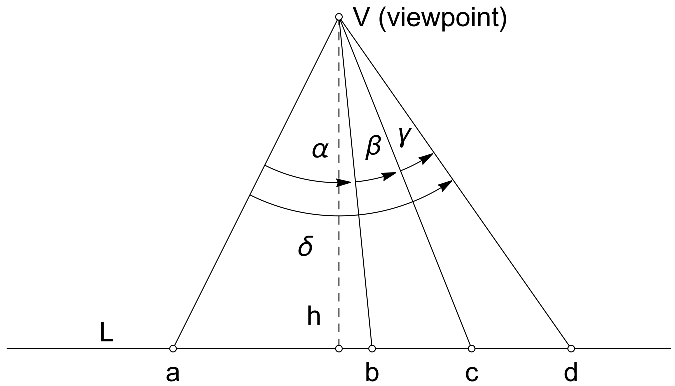

Figure 1. Viewing angles of points on ℂ

14. The figure below [p. 185] shows four collinear points a, b, c, d, together with the (necessarily coplanar) light rays from those points to an observer. Imagine that the collinear points lie in the complex plane, and that the observer is above the plane looking down. Show that the cross-ratio [a, b, c, d] can be expressed purely in terms of directions of these light rays; more precisely show that

$$[a, b, c, d] = -\frac{\sin\alpha \sin\gamma}{\sin\beta \sin\delta}.$$

Suppose an observer now does a perspective drawing on a glass "canvas plane" C (arbitrarily positioned between himself and ℂ). That is, for each point p in ℂ, he draws a point p' where the light ray from p to his eye hits C. Use the above result to show that although angles and distances are both distorted in his drawing, cross-ratios of collinear points are preserved: [a', b', c', d'] = [a, b, c, d].

Rewrite the cross-ratio by analogy to the equation on p. 154:

$$[a, b, c, d] = \frac{(a-b)(c-d)}{(a-d)(c-b)} \qquad (1)$$

The construction is somewhat mixed in terms of mathematical objects. The line L and the collinear points a, b, c, d are on the complex plane, but the viewpoint is above the plane. This makes it problematic to calculate the lengths of sides from points, because one can't easily add or subtract the collinear points in the complex plane to or from the viewpoint, which must be in three-space. Vasco has found a way around this problem which enables consideration of the lines of projection as lengths or absolute values. Needham has constructed the exercise over the complex plane. Using $z = |z| e^{i\theta}$, and recognizing that all the sums of collinear points have the same direction, which is plus or minus the direction of L, rewrite (1):

$$\frac{(a-b)(c-d)}{(a-d)(c-b)} = \frac{|a-b|e^{i\theta}|c-d|e^{i\theta}}{|a-d|e^{i\theta}|c-b|e^{i(\theta-\pi)}} = \frac{|a-b||c-d|e^{i\pi}}{|a-d||c-b|} = -\frac{|a-b||c-d|}{|a-d||c-b|} \qquad (2)$$

Identify four triangles by their peak angles: α, β, γ, δ, as in Figure 1. The values in the absolute brackets in (2) are the lengths of the bases of the triangles. Since there is only one viewpoint, the height h of all the triangles is equal, so the lengths of the bases are determined only by the angles between the sides drawn from the viewpoint, V. To find an equation that expresses the cross-ratio in terms of sines, an equation is needed that makes use of the sine in a triangle. An equation for the area of a triangle which satisfies this need is A = (1/2) r s sin(θ), where r and s are the lengths of the sides that intersect at θ. From another equation A = (1/2) bh, we know that the area is proportional to the base and the height, but the height in this case is constant, so we need only consider area as proportional to the base. That means one can rewrite (2) by substituting areas for bases. In the following equations, the square brackets indicate the length of the line from a to V, b to V, etc.

$A_\alpha =$ (1/2) [aV] [bV] sin α
$A_\beta =$ (1/2) [bV] [cV] sin β
$A_\gamma =$ (1/2) [cV] [dV] sin γ
$A_\delta =$ (1/2) [aV] [dV] sin δ

$$-\frac{|a-b|\,|c-d|}{|a-d|\,|c-b|} = -\frac{A_\alpha A_\gamma}{A_\delta A_\beta} - \frac{(1/2)\,[aV]\,[bV]\ \sin\,\alpha\,(1/2)\,[cV]\,[dV]\ \sin\,\gamma}{(1/2)\,[aV]\,[dV]\ \sin\,\delta\,(1/2)\,[bV]\,[cV]\ \sin\,\beta} = -\frac{\sin\alpha\sin\gamma}{\sin\delta\sin\beta}$$

This is what we wanted.

In the second part of the exercise, the glass plane C is placed between $\mathbb{C}$ and the viewpoint. Therefore, the actual distances on the glass between the points a', b', c', d' depend on the orientation (tilt, slant) of the glass away from the horizontal complex plane. Angles of intersecting lines on the complex plane change when projected onto the intervening glass. Let L' be the line L projected to the glass. One can "drop" a line segment from the viewpoint to L' intersecting it at a right angle. Let the length of this line be "h'"; it is the height of triangles plotted from the viewpoint to the collinear points projected onto L'. Height is constant and the angles at the viewpoint are constant, so one can follow the same procedure as above applied to the points on L' to obtain the cross-ratio.

[a', b', c', d'] = - $\frac{\sin\alpha\ \sin\gamma}{\sin\beta\ \sin\delta}$ = [a, b, c, d]