Needham, Chapter 3, Exercise 13 (ii), p. 184.
Gary Palmer, June 25, 2021, 11:50 am PDT

(ii) "Let A, B, and C be three spheres (not necessarily of equal size) all touching one another. As in the previous part, let S_1, S_2, ... be a string of spheres (now of unequal size) touching one another successively, and all touching A, B, C. Astonishingly (compare previous exercise), S_6 will always touch S_1, forming a closed 'necklace' of six spheres interlocked with A, B, C. Prove this by first applying an inversion centred at the point of contact of A and B, then appealing to part (i)." ***

Preliminary remarks: It is a project in itself to plot the construction described in this exercise (Figure 2, left and Figure 3). Beginning with the construction in Figure 1 (left), the spheres are inverted in K_1, which is another sphere centred on the X axis (Figure 1). The K_1 inversion preserved all the touch points but changed the sizes and centres of A, B, and C. A second inversion in sphere K_2, which has an arbitrary centre and radius (Figure 2) gives the construction described in the exercise and shown in Figures 2 (left) and 3 with different size spheres A', B', and C' and chain spheres not necessarily of equal size. The chain is still closed and all the spheres are touching all the other spheres.

Since K_3 touches the intersection of A' and B', on inversion in K_3 one sees parallel planes A'' and B'' touching either side of the chain with C'' in the center of the chain and equal in size to the chain spheres. We arrive at the configuration described in Part (i) except that instead of the table top, we see two planes, A'' and B''. The plot in Figure 4 has the configuration described in (ii) "S_6 will always touch S_1, forming a closed 'necklace' of six spheres interlocked with A, B, C,". We see this as the result of the one-to-one inside-outside mapping of inversion in the circle generalized to the sphere and the anticonformal properties of inversion which assure that angles (and thereby points of tangency) are preserved, while distances and orientations of angles may vary.

*touching: two objects "touch" if they have a common tangent line.
**A closed chain of six spheres of equal size lying on a table top, all touch a central sphere of the same size.

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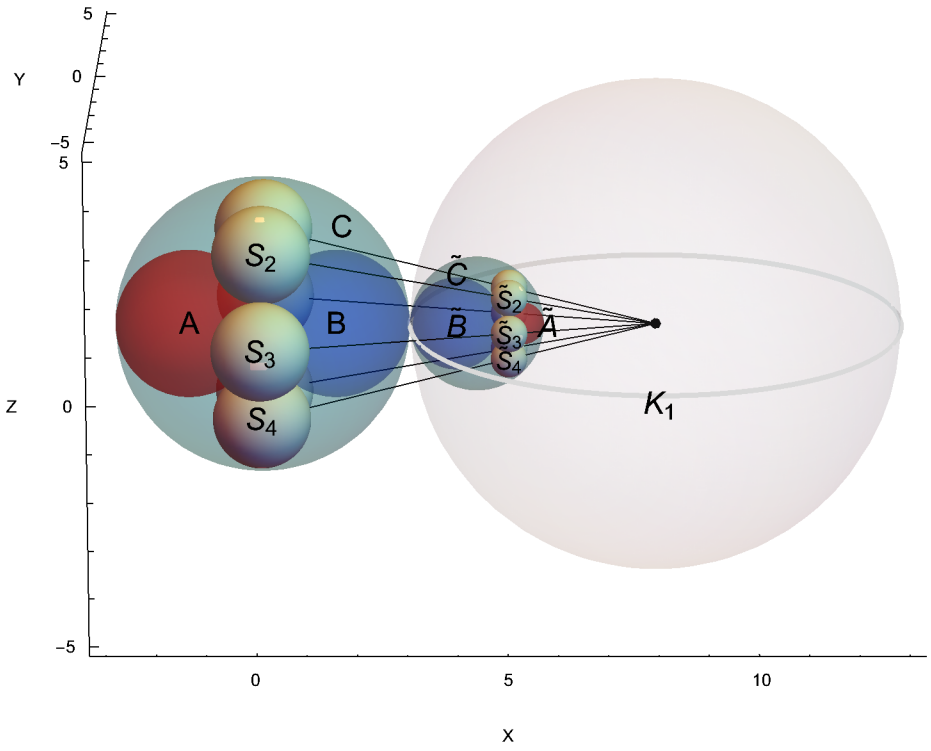

Figure 1.
Closed chain of white spheres S_i touches spheres A (red), B (blue), and C (teal).
Sphere of inversion K_1 outlined by great circle.
K_1 touches B and C. $\tilde{A}$ and $\tilde{B}$ are unequal. All spheres touch.

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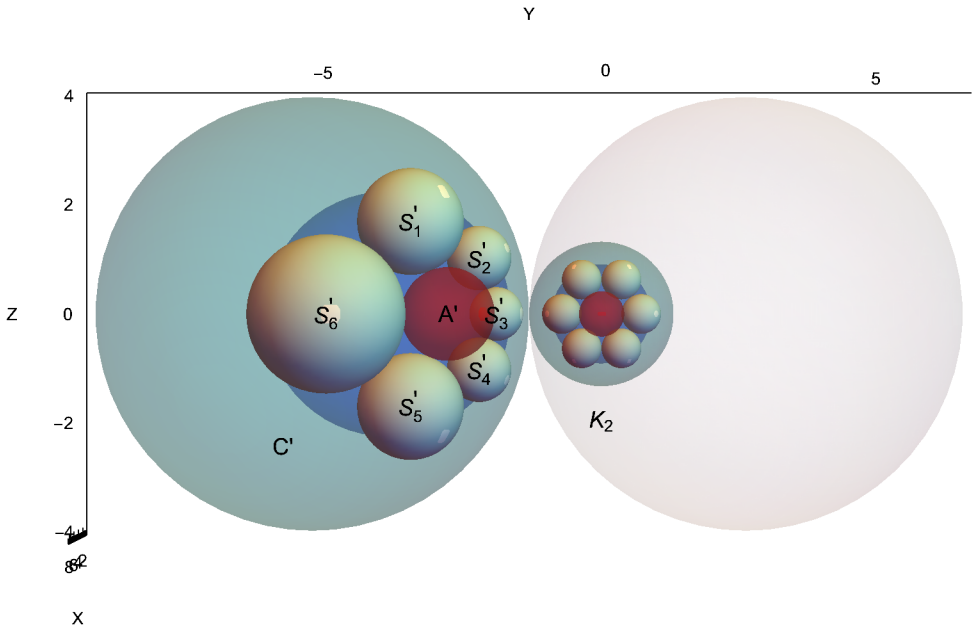

Figure 2. After inversion in K_2, C' touches K_2.
All spheres touch and chain spheres S_i' are unequal.

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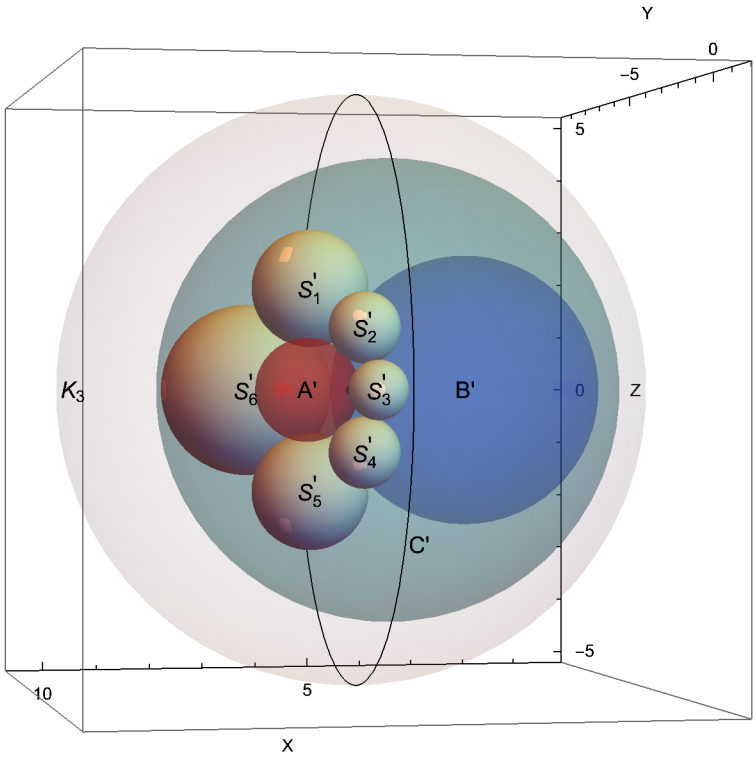

Figure 3. K_3 in gray w. great circle centred at A' ⋂ B'. A' and B' should become planes A'' and B'' (not shown).

Out[•]=

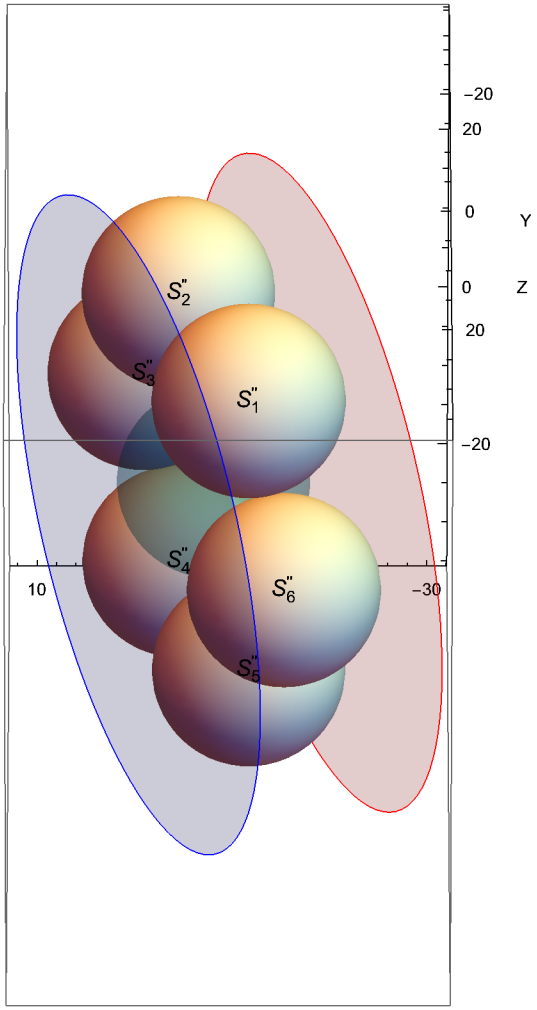

Figure 4. K_3 (not shown) centred at A' ⋂ B'.
A' and B' (not shown) become planes A'' (pink) and B''(light blue).
A'' and B'' touch all the spheres inverted in K_3.
Chain spheres are equal size and closed.
C'' (teal) is centred and sized correctly and touches chain spheres.