Needham, Chapter 3, Exercise 12, p. 184.
G. Palmer, May 31, 2021, 6 pm PDT

BRIEF RESTATEMENT OF THE PROBLEM

If a chain C_i of touching circles that also touch two non-intersecting, non-concentric circles A and B closes under a Möbius transformation for one choice of C_1, then it closes for every choice of C_1, and the resulting chain always contains the same number of touching circles. Explain this using the result of Ex. 10.

Result of Ex. 10: *Any two non-intersecting, non-concentric circles can be mapped to concentric circles by means of a suitable Möbius transformation* (p. 183).

Möbius transformations are one-to-one and conformal (p. 176). From that we can deduce that an appropriate M can transform a set of concentric circles into a non-concentric set of circles.

Since Möbius transformations are one-to-one, a transformation M sends circle to circle. Hence, the resulting chain contains the same number of circles.

Since Möbius transformations are conformal, they preserve magnitudes and orientation of angles. A closed ring of touching circles that all touch two non-concentric circles has equal angles at all the points of tangency: 0 or π, depending on whether the direction of orientation is considered. In the transformation, these angles are all preserved. If the circles in the chain were changed to overlap, their tangent angles would be destroyed and the chain would not close. If more circles were added, tangent angles would be added and others would be distorted. If circles were subtracted, tangent angles would be destroyed and others would be distorted. But suppose we begin with a concentric frame. Since the frame is concentric, all the chain circles have the same radius. From the result of Ex. 10, one can map this to a non-concentric frame with a closed chain consisting of the same number of circles, all touching A and B. One can change circle C_1 in the non-concentric frame without disturbing the framework by rotating the the complex plain about the centre of the concentric circles. Changing C_1 in the concentric frame entails change to C_1 and all the C_i in the non-concentric nested frame.

From these facts one can deduce that no matter the starting place, and no matter the number of circles in the chain, if the chain closes for one choice of C_1, it closes for every choice of C_1.

Another implication of the fact that MT's are one-to-one is that they have an inverse. To demonstrate the exercise, we begin with a closed chain of equal-sized circles touching two concentric circles. We use an M to send the construction to a closed chain of unequal-sized circles still touching two non-concentric circles. We know that this non-concentric frame can be sent back to the concentric frame with the inverse of M, so one could begin the exercise with the result in (3), apply M inverse to revert to (1), change C_1, and reapply M to obtain the result in (6). [Mathematica would not permit me to typeset the inverses here.]

For this exercise, we construct M by the composition of two non-conformal inversions in the circle $M = \mathcal{I}_{K_2} \circ \mathcal{I}_{K_1}$. Then ($M$ inverse) = $\mathcal{I}_{K_1} \circ \mathcal{I}_{K_2}(w)$. The transformation must be carefully constructed to achieve the desired result. Here, we use the transformations from Exercise 11 in reverse. That is, $M = \mathcal{I}_{K_p} \circ \mathcal{I}_{K_g}(z)$. Figures (1) to (3) show a transformation from a concentric frame to a non-concentric frame. Notice that C_1 in (1) is centred on the real line. In Figure (4), we rotate frame (1) by $\pi/6$. This provides a new C_1. In (5) and (6), M is applied to the changed frame in (4). In (6) we see that the chain is still closed, the number of circles in the chain remains the same, and the radii of A and B are equal to the radii of A and B in (3).

To follow the transformations it helps to realize that A' and B' are centred at h' (the origin in this case) and that B' touches g, the center of the circle of inversion K_g (Figure (1)). In Figure (2), $\tilde{A}$ and $\tilde{B}$ are orthogonal to $K_{\tilde{q}}$ and $\tilde{B}$ passes through $\tilde{q}$. This has implications for the radius of $K_{\tilde{q}}$. In Figure (3), B touches the centre of K_p and it also touches $q = \mathcal{I}_{K_p}(\tilde{q})$. The fact that this exercise can be demonstrated by reversing the transformations in Exercise 11 supports Vasco's reinterpretation of that exercise.

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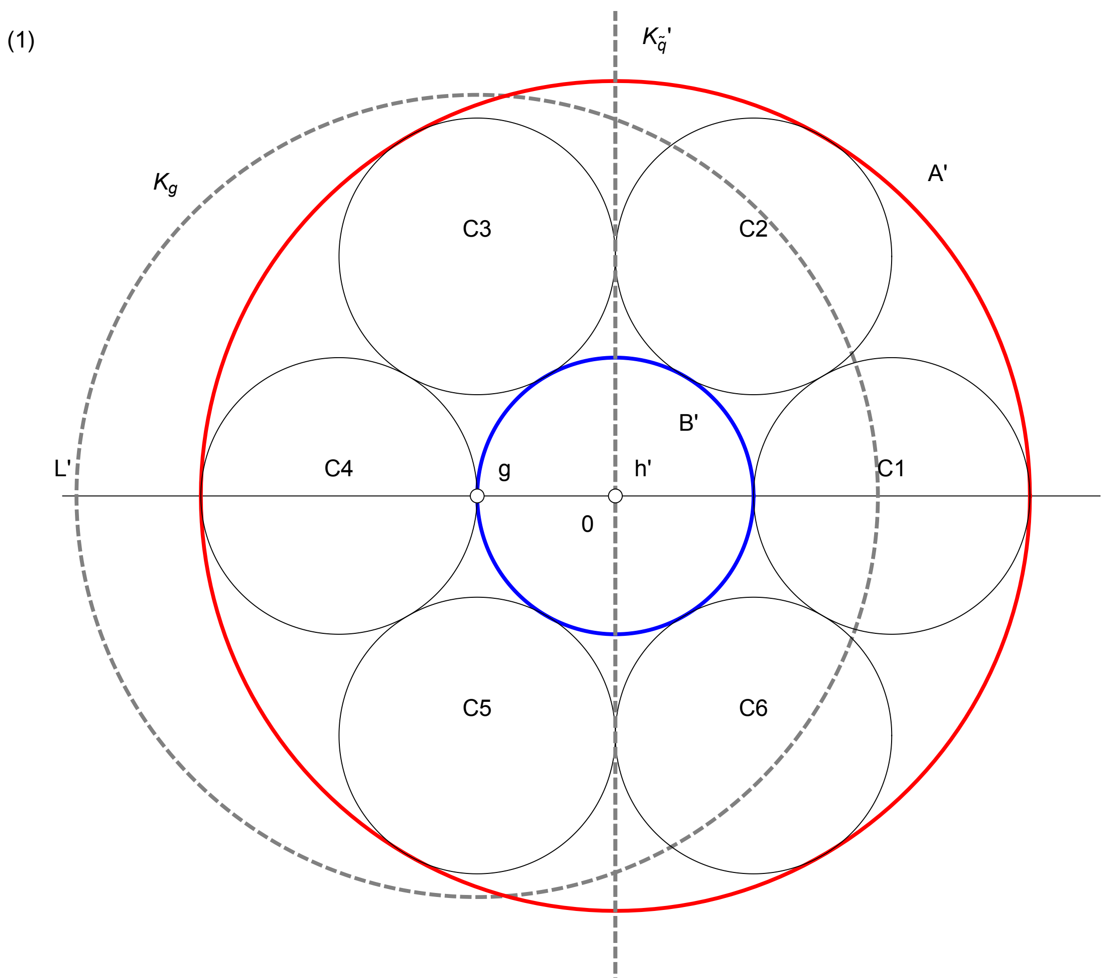

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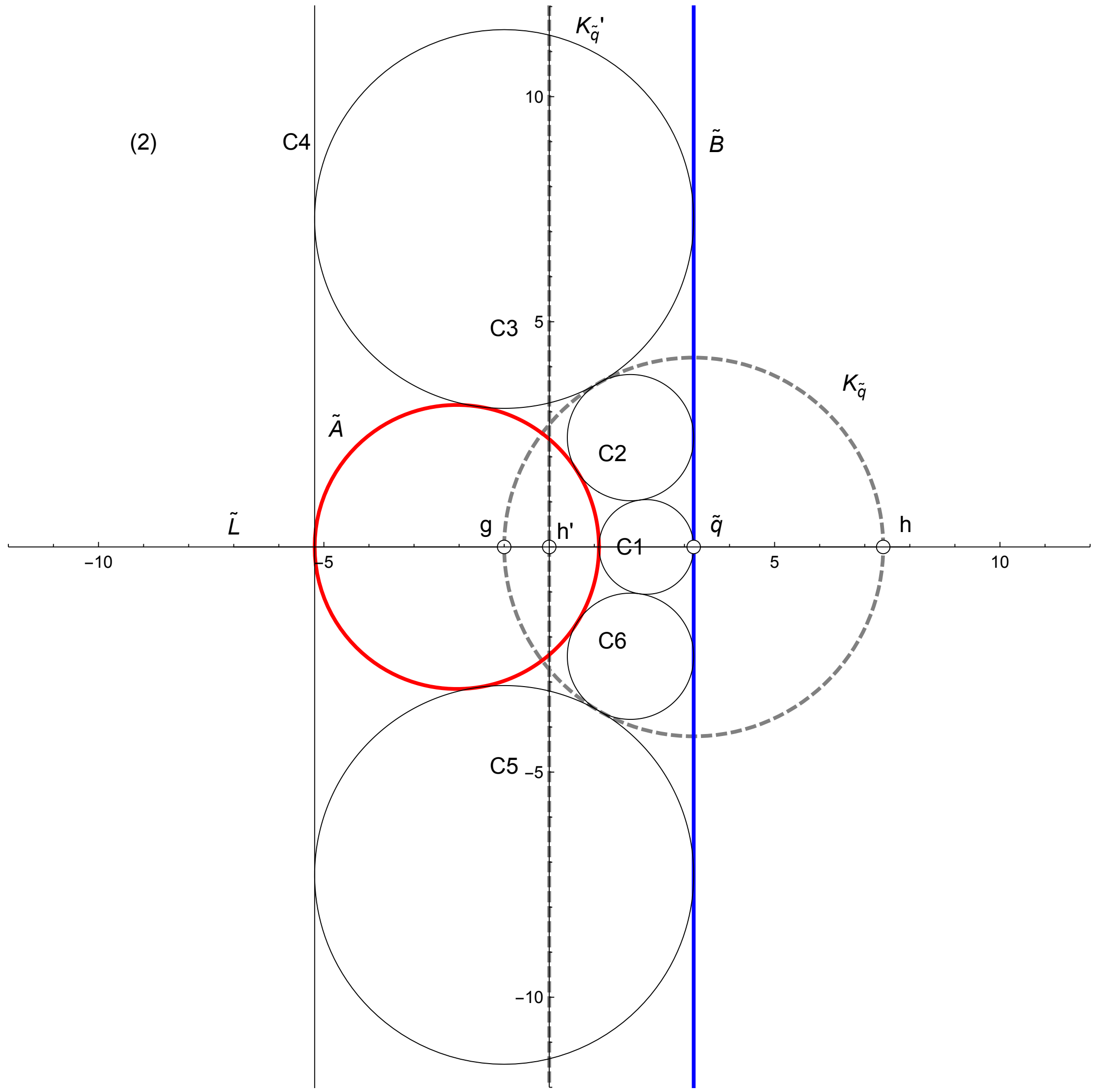

(2)
C4
$K_{\tilde{q}}'$
$\tilde{B}$
C3
$K_{\tilde{q}}$
$\tilde{A}$
C2
$\tilde{L}$
g
h'
C1
$\tilde{q}$
h
C6
C5
−10
−5
5
10

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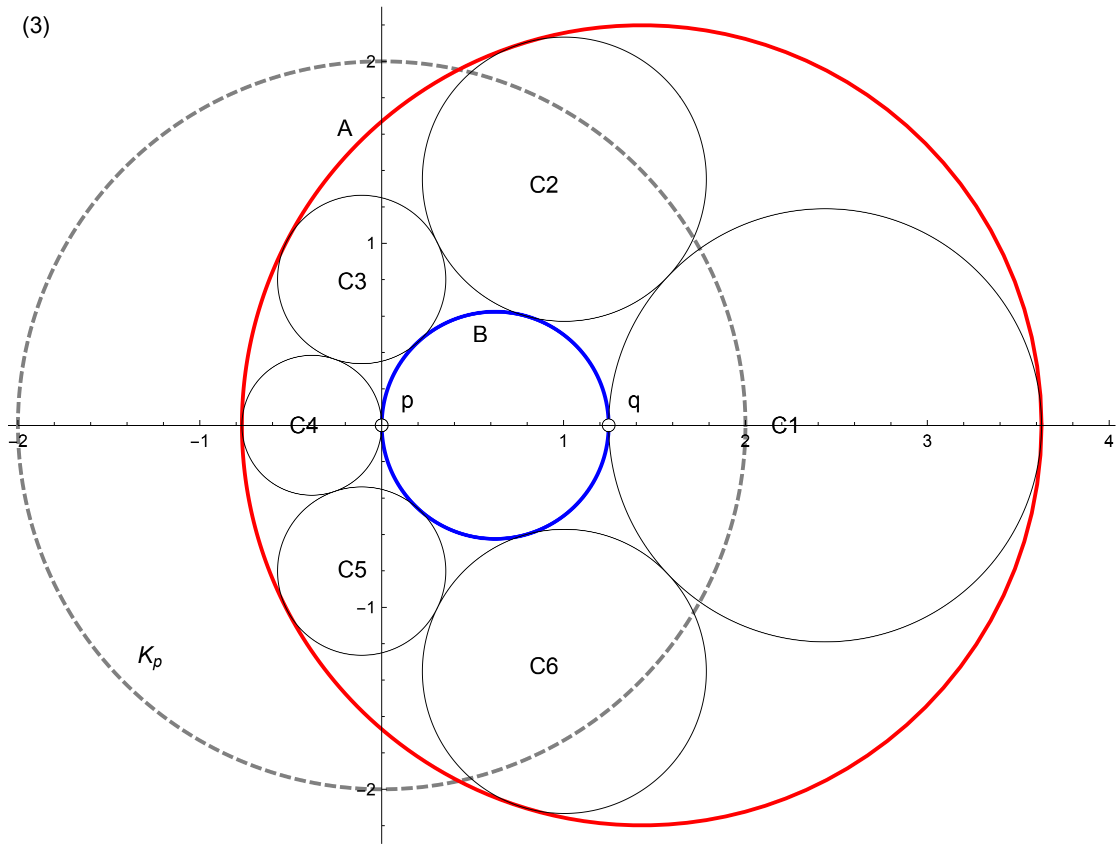

(4)

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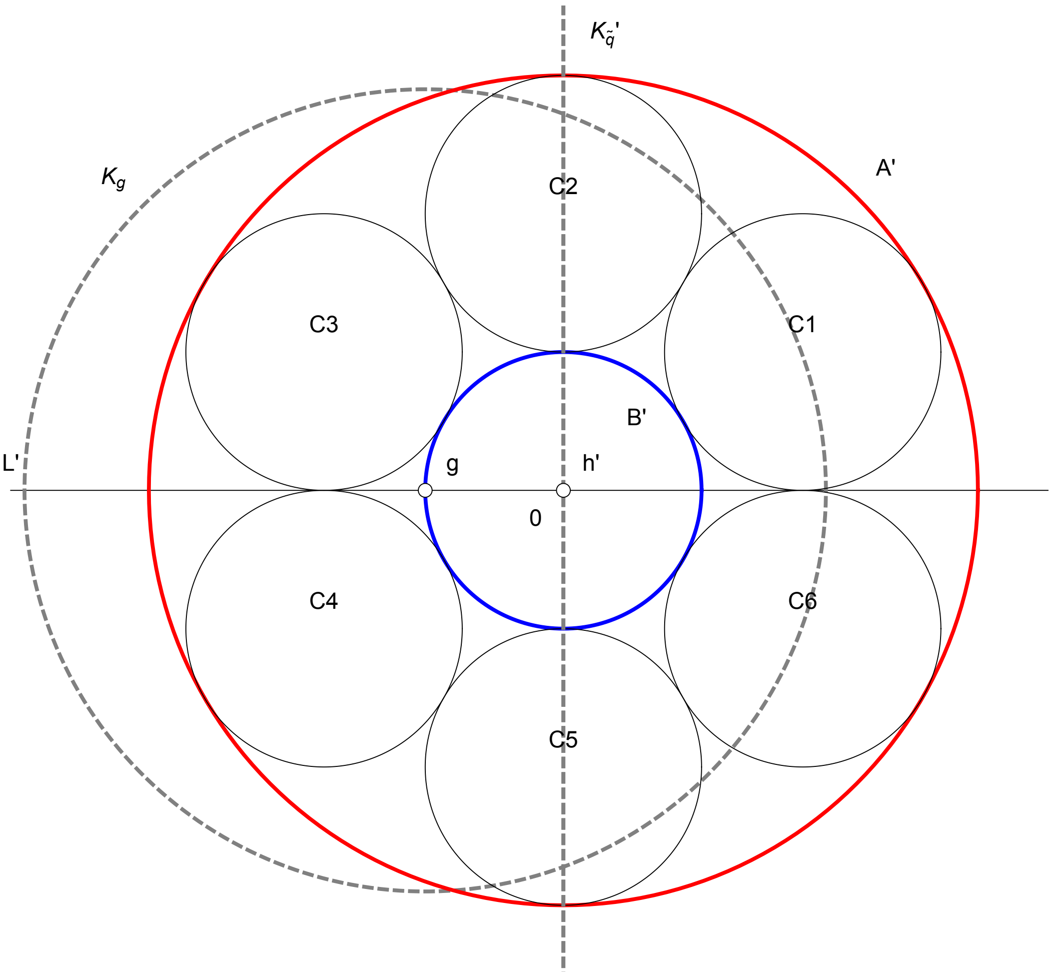

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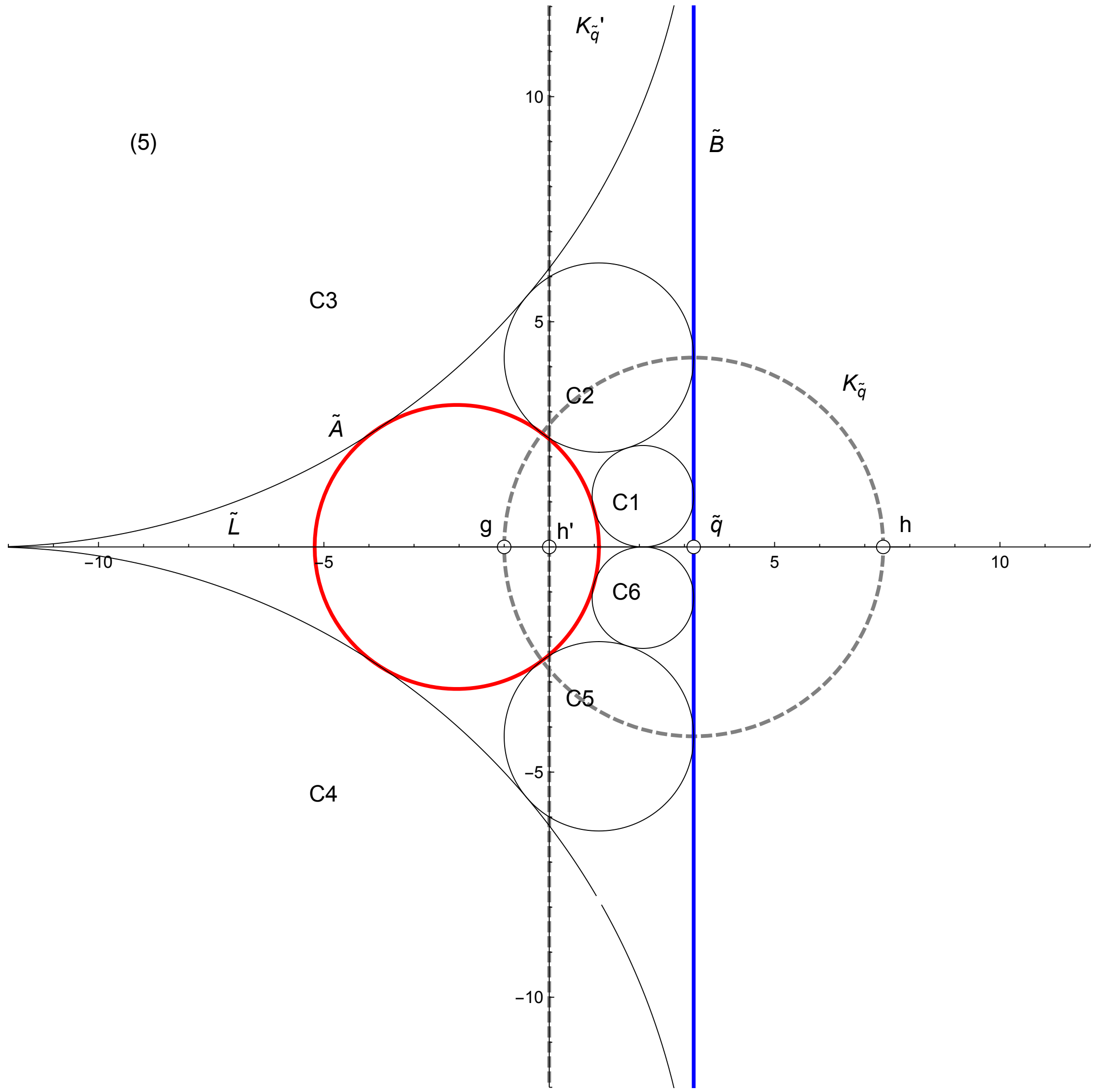

(5)
$K_{\tilde{q}}'$
10
5
−5
−10
5
10
$\tilde{B}$
C3
C2
$K_{\tilde{q}}$
$\tilde{A}$
C1
$\tilde{L}$
g
h'
$\tilde{q}$
h
C6
C5
C4

(6)

Out[ ]=