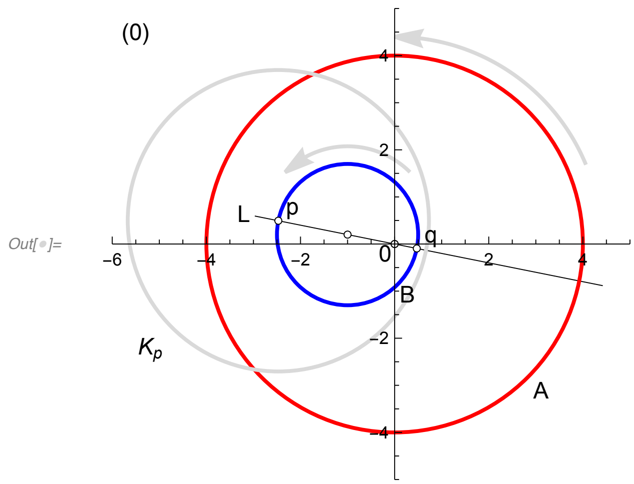
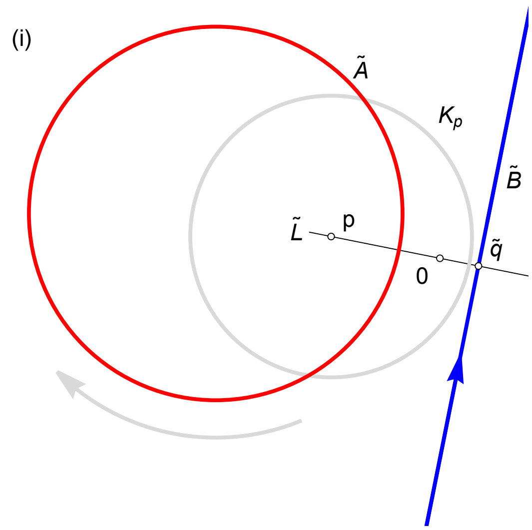
Needham, Chapter 3, Exercise 11, p. 183.
Gary Palmer, May 31, 2021, 10:45 am PDT





11 This exercise yields a more intuitive proof of the result of the previous exercise. Using different colors for each, draw two non-intersecting , non-concentric circles, A and B, then draw the line L through their centres. Label as p and q the intersection points of B with L.

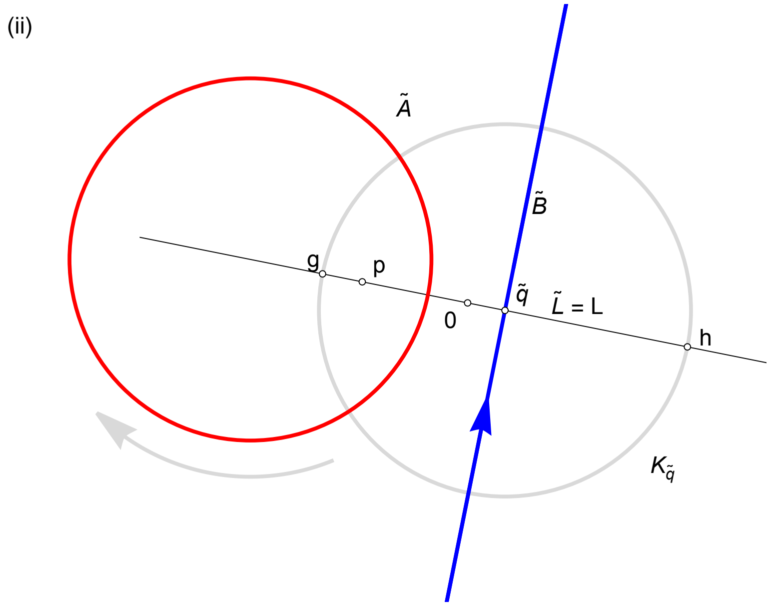
See Figure (0).

(i) Using corresponding colors, draw a fresh picture showing the images $\tilde{A}$, $\tilde{B}$, $\tilde{L}$, $\tilde{q}$ of A, B, L, q under inversion in any circle centred at p. To get you started, note that $\tilde{L} = L$.

[At this point, Mathematica begins to fail to display tildes, as sometimes happens with files that have undergone numerous revisions, so I substitute A~, B~, etc. for the overline tilde A' and B', etc. for "A prime", "B prime".]

The inversion* reverses the left to right order of A and B on L to that of B~ and *A*~ on *L*~ (Figure [i]). *B*~ is the infinite circle because B touches p (Figure [0]). q~ is a point in B~ (Figure [i]). The arrows indicate orientation. Clockwise is negative. Anti-clockwise is positive.

Out[ ]=



(ii) Now add to your figure by drawing the circle K, centred at $\tilde{q}$, which cuts $\tilde{A}$ at right angles, and let g and h be the intersection points of K and L.

See figure (ii). How can one arrange for K ($K_{\tilde{q}}$ in figure ii)** to be both orthogonal to A~ and centred at q~? From Exercise 10, we know it happens if a point in A~ is symmetric in both A~ and B~, which are non-concentric and non-intersecting. Then we can plot $K_{\tilde{q}}$ as an analog to one of the C_1 circles in the discussion on p. 163 pertaining to [29a]. One can regard A~ and B~ as analogs to the C_2 circles, because g and h will be symmetric in both. Since B~ is an infinite circle, inversion in B~ is the reflection $\mathcal{R}_{\tilde{B}}$(z). We wish to plot g = $\mathcal{R}_{K_{\tilde{q}}}(h)$ and h = $\mathcal{I}_{K_{\tilde{q}}}(g)$. This guarantees that $K_{\tilde{q}}$ is orthogonal to A~. We could find the two points where K and A~ intersect at right angles by plotting the intersection of rays tangent to the circles, but it seems preferable to work with the concept of inversion.

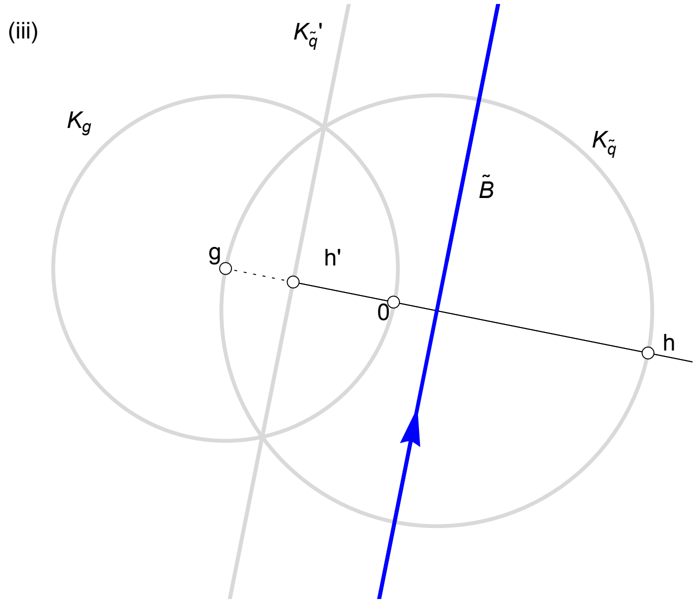
Let m and n be the two points where A intersects L. Then m~ and n~ are the two points where A~ intersects L~. Adapting the equation on p. 163 to a vector format, one can calculate the radius $r_{K_{\tilde{q}}}$ of $K_{\tilde{q}}$.

$$r_{K_{\tilde{q}}} = \sqrt{|\, m\sim - q\sim || n\sim - q\sim \,|}$$

We can now plot $K_{\tilde{q}}$ centred at q~. Points g and h are the points where $K_{\tilde{q}}$ intersects L. We find empirically that g and h reflect in B~ and they are inversions in A~.

Notice also that both A~ and B~ are orthogonal to $K_{\tilde{q}}$. This is important in the explanation of concentricity in part (iv). When we apply inversion in a circle to A~, B~, and $K_{\tilde{q}}$, we can expect that A' and B' will be orthogonal to K_q~' (K_\tilde{q}^\prime). The only way that the circles A' and B' can be concentric is if both are orthogonal to a line through their common centre. Then K_q~' must be that line through their centre.

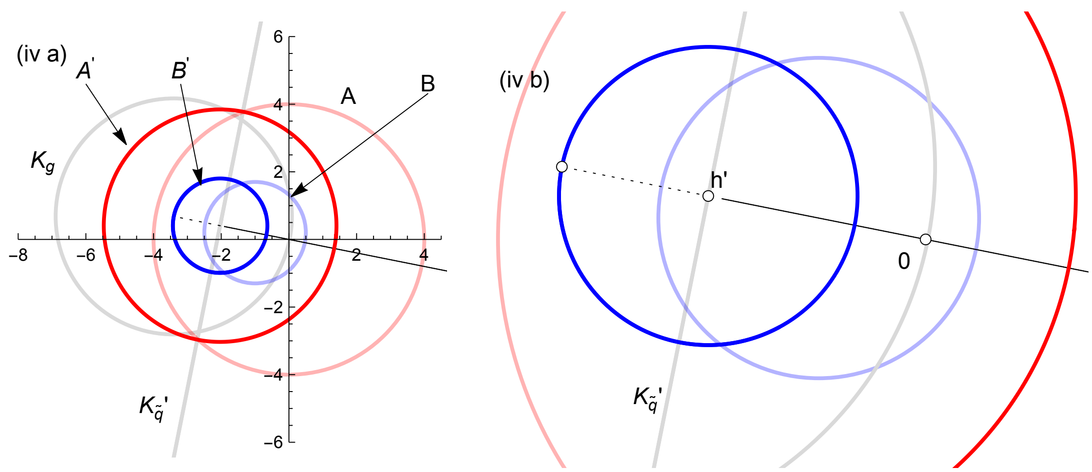
Out[•]=



(iii) Now draw a new picture showing images K' [$K_{\tilde{q}}$ '], L', h' of K [$K_{\tilde{q}}$], L, h under inversion in any circle centred at g.

Since $K_{\tilde{q}}$ intersects L at g by construction, $K_{\tilde{q}}$' is an infinite circle orthogonal to L' extended, shown in Figure (iii) (See also, p. 127). Since h is in $K_{\tilde{q}}$, h’ must be in K_q~'. By the last sentence of part (ii), h’ lies on L’, so it must be the common centre of A’ and B’.

Out[•]=



(iv) By appealing to the anticonformal nature of inversion, deduce that $\tilde{A}$, $\tilde{B}$ are concentric circles centred at h'.

Vasco has shown that Needham's question can be answered by treating part (iii) not as a result to be compared, but as the model for a second inversion that can be applied to both A~ and *B*~. That is, when we invert both A~ and *B*~ in K_g, we obtain concentric circles *A*' and *B*'. Needham wrote *Since the composition of two inversions is a Möbius transformation, you have proved the result of the previous exercise*. In Figure (ii) we see that A~ and *B*~ have undergone a single inversion. This remains true in Figure (iii), although A~ is not shown. In Figure (iv) we reintroduce *A*~ and subject both *A*~ and B~ to a second inversion, inversion in K_g. In part (ii) we pointed out that A~ and *B*~ are orthogonal to $K_{\tilde{q}}$ in Figure (ii), so *A*' and *B*' must be orthogonal to $K_{\tilde{q}}$'. This is because inversion in the circle is anticonformal, which entails that right angles are preserved, even though their orientation is reversed. *A*', *B*', and $K_{\tilde{q}}$' were all created by inversion in K_g.

To sum up the process: Inversion in K_p sends B to an infinite circle B~ and A to a circle A~ . Then we plot a circle orthogonal to A~ and centered at the intersection of B~ and L~. This makes it the new circle orthogonal to both A~ and B~. We call the intersection of of B~ and L~ " q~". To make the circle $K_{q\sim}$ orthogonal to A~, we need the radius of $K_{q\sim}$. For that we use the square root of the product of the distances to the intersections of A~ with L~, which works because $K_{q\sim}$ is orthogonal to A~ by construction. Then, to make A' and B' concentric, we make sure that $K_{q\sim}$ touches the centre of the circle to be constructed for the final inversion. We place the center g of K_g at the intersection of $K_{q\sim}$ with L~. Inversion in K_g sends $K_{q\sim}$ to the infinite circle $K_{q\sim}$'. Since inversion in the circle preserves orthogonality, A' and B' are orthogonal to $K_{q\sim}$' and to L', and therefore, they must be concentric. Now we might rewrite (iv) as *By appealing to the anticonformal nature of inversion, deduce that A' and B'* [not A~ and *B*~] *are concentric circles centred at h'*. We see that h inverted in K_g maps to the intersection of $K_{\tilde{q}}$' with L'. Note that A' and B' are somewhat smaller than A and B.

*By inversion is meant $\mathcal{I}_K(z)$, inversion in a circle K, where $\mathcal{I}_K(z) = \frac{R^2}{\bar{z}-\bar{q}} + q$, R is the radius and q is the centre.

**I find it helps to keep the circles straight by identifying them by their centers. I use K_p, $K_{\tilde{q}}$, and K_g.