Needham, Chapter 3, Ex. 10, p. 183.

G. Palmer, April 30, 1:30 pm PDT

10. The aim of this question is to understand the following result:

Any two non-intersecting, non-concentric circles can be mapped to concentric circles by means of a suitable Möbius transformation.

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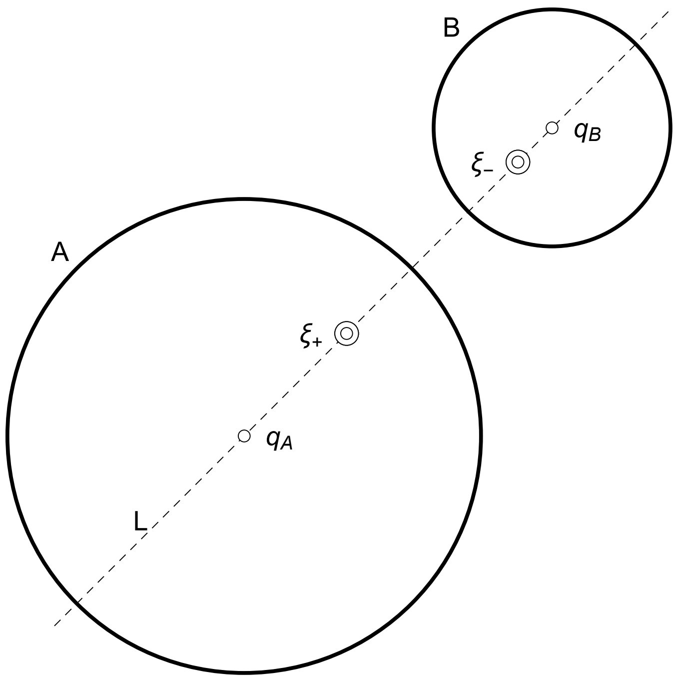

Figure 1. Points symmetric in two circles

(i) If A and B are the two circles in question, show that there exists a pair of points $\xi_\pm$ that are symmetric with respect to both A and B.

We use formula (4) from page 125 for inversion in the circle, but instead of finding a new circle of inversion, we define $\mathcal{I}_A(z)$ and $\mathcal{I}_B(z)$ using the radii and centres of circles A and B (Figure 1).

$I_K(z) = \frac{R^2}{\bar{z} - \bar{q}} + \text{q}$ (4) (the z has an overline)

If a point z is symmetric with respect to both A and B, we can write:

$\mathcal{I}_A(z) = \mathcal{I}_B(z) = \tilde{z}$ (1)

Substitute appropriately into (4).

$$\frac{R_A{}^2}{\bar{z}-\bar{q}_A} + q_A = \frac{R_B{}^2}{\bar{z}-\bar{q}_B} + q_B \qquad (2)$$

Vasco here introduces a simplification by conjugating both sides.

$$\frac{R_A{}^2}{z-q_A} + \bar{q}_A = \frac{R_B{}^2}{z-q_B} + \bar{q}_B$$

Multiply through by $(z - q_A)(z - q_B)$.

$$R_A{}^2(z - q_B) + \bar{q}_A(z - q_A)(z - q_B) = R_B{}^2(z - q_A) + \bar{q}_B(z - q_A)(z - q_B)$$

$$(\bar{q}_A - \bar{q}_B)\,(z - q_A)\,(z - q_B) + R_A{}^2(z - q_B) - R_B{}^2(z - q_A) = 0$$

Divide through by $(\bar{q}_A - \bar{q}_B)$.

$$(z - q_A)\,(z - q_B) + [R_A{}^2(z - q_B) - R_B{}^2(z - q_A) / (\bar{q}_A - \bar{q}_B)] = 0$$

$$z^2 - q_A\,z - q_B\,z + q_A\,q_B + [R_A{}^2(z - q_B) - R_B{}^2(z - q_A)] / (\bar{q}_A - \bar{q}_B) = 0$$

Collect like powers.

$$z^2 - (q_A + q_B)\,z + q_A\,q_B + (R_A{}^2\,z - R_A{}^2\,q_B - R_B{}^2\,z + R_B{}^2\,q_A)/(\bar{q}_A - \bar{q}_B) = 0$$

$$z^2 + [-(q_A + q_B) + (R_A{}^2 - R_B{}^2)/(\bar{q}_A - \bar{q}_B)]\,z + q_A\,q_B + (R_B{}^2\,q_A - R_A{}^2\,q_B)/(\bar{q}_A - \bar{q}_B) = 0$$

Apply the quadratic equation, $z = (-b \pm \sqrt{b^2 - 4\,ac}\,)/2a$.

$a = 1$
$b = -(q_A + q_B) + (R_A{}^2 - R_B{}^2)/(\bar{q}_A - \bar{q}_B)$
$c = q_A\,q_B + (R_B{}^2\,q_A - R_A{}^2\,q_B)/(\bar{q}_A - \bar{q}_B)$

$$z = \frac{(q_A+q_B) - \left(R_A{}^2 - R_B{}^2\right)/\left(\bar{q}_A - q_B\right) \pm \sqrt{\left(-(q_A+q_B) + \left(R_A{}^2 - R_B{}^2\right)/\left(\bar{q}_A - q_B\right)\right)^2 - 4\left(q_A\,q_B + \left(R_B{}^2\,q_A - R_A{}^2\,q_B\right)/\left(\bar{q}_A - \bar{q}_B\right)\right)}}{2}$$

Since the quadratic has two values, this shows that there are two points that are symmetric in both A and B. The solutions to z are the points $\xi_\pm$ in Figure 1. To check the result, both formulas for the inverse were used to plot ξ_+ and ξ_-.

One can simplify and greatly reduce the difficulty of solving for $\xi_\pm$ by aligning the circles to the real line. Since $\bar{z} = z$ on the real line, one can dispense with conjugates.

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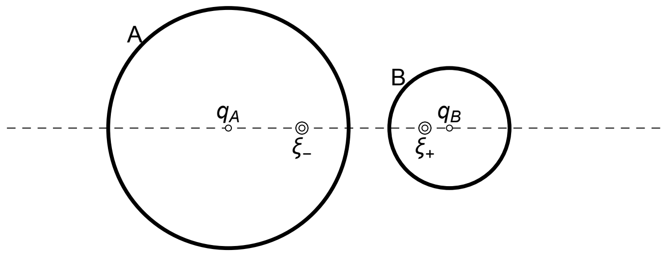

Figure 2. Inverse points in two circles on real line

Align the centres of A and B on the real axis and place q_A at the origin. Then, if the pair of inverse points can be found, the $\mathbb{C}$ plane can be rotated about the origin by any angle and translated by any vector. The rotation plus translation preserves the circles and their inversions. To keep the typesetting simple, we will begin by calculating ξ_-. Since the centres of both circles are on the real line, $\xi_\pm$ and q_B are real numbers. We can write a new version of equation (2).

$$\frac{R_A{}^2}{\xi_+} = \frac{R_B{}^2}{\xi_+ - q_B} + q_B$$

Multiply through by $\xi_+(\xi_+ - q_B) = (\xi_+^2 - q_B\,\xi_+)$

$$R_A{}^2(\xi_+ - q_B) = R_B{}^2\xi_+ + q_B\xi_+(\xi_+ - q_B)$$

$$R_A{}^2\xi_+ - R_A{}^2\,q_B = R_B{}^2\xi_+ + q_B\xi_+^2 - q_B^2\,\xi_+$$

Rearrange.

$$q_B\xi_+^2 + R_B{}^2\,\xi_+ - q_B^2\,\xi_+ - R_A{}^2\,\xi_+ + R_A{}^2\,q_B = 0$$

Divide through by q_B.

$$\xi_+^2 + \frac{(R_B{}^2 - q_B^2 - R_A{}^2)}{q_B}\,\xi_+ + R_A{}^2 = 0$$

$$\xi_+ = \frac{-\frac{(R_B{}^2 - R_A{}^2 - q_B^2)}{q_B} \pm \sqrt{\left(\frac{(R_B{}^2 - R_A{}^2 - q_B^2)}{q_B}\right)^2 - 4\,R_A{}^2}}{2} \quad \text{(by the quadratic equation)}$$

Then $\xi_- = \mathcal{I}_A(\xi_+) = \mathcal{I}_B(\xi_+)$. With this construction on the real line, we also find from p. 124, with $q_A = 0$ that

$$R_A^2 = (\xi_+ - q_A)\,(\xi_- - q_A) \Longrightarrow \xi_- = R_A^2/\xi_+$$

One can see in Figure 2, that these calculations produce the desired result. The values of ξ_+ and ξ_+ are checked with $\xi_- = \mathcal{I}_A(\xi_+) = \mathcal{I}_B(\xi_+)$ and $\xi_+ = \mathcal{I}_A(\xi_-) = \mathcal{I}_B(\xi_-)$. One can see that all the values fall on one of the same two points.

Now having chosen two non-intersecting, non-concentric circles A and B of arbitrary sizes and locations, one can place them in this construction with the movement $R_0^{-\theta} \circ T_{-q_A}(z)$, where θ is the angle between the the real axis and the segment connecting q_A and q_B. Then having found ξ_+ and ξ_+, one can apply the reverse movement to $\mathbb{C}$ with $T_{q_A} \circ R_{q_A}^{\theta}(z)$ to restore the circles to their original positions. We have shown *that there exists a pair of points $\xi_\pm$ that are symmetric with respect to both A and B*, and we have demonstrated the solution with an actual instance.

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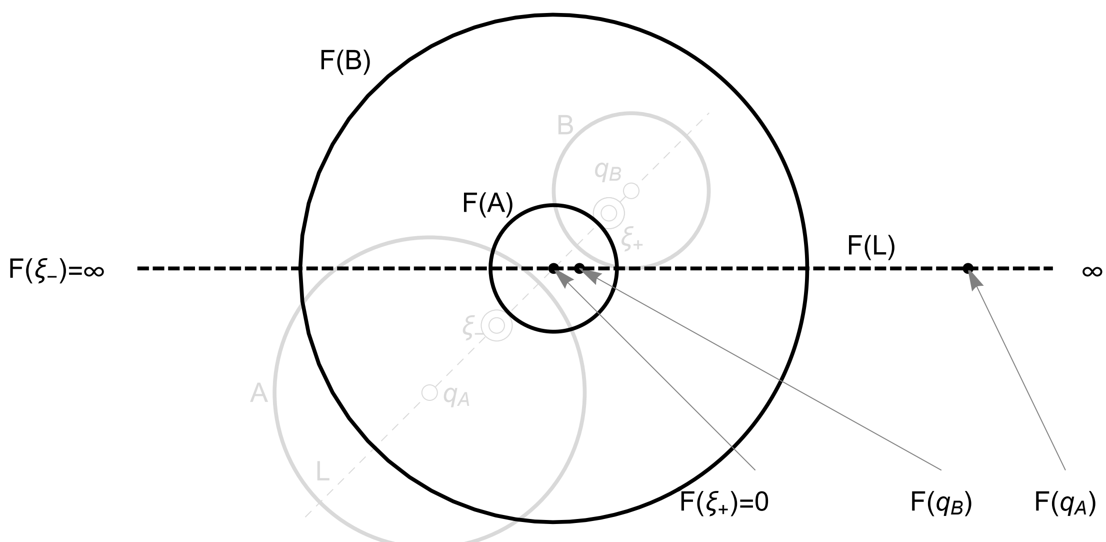

Figure 3. F(z) = (z−ξ_+)/(z−ξ_-) applied to A and B superimposed on Figure 1

(ii) Deduce that if F(z) = (z - ξ_+) / (z − ξ_-), then F(A) and F(B) are concentric circles, as was desired.

F(z) is a Möbius transformation. F(A) and F(B) are circles because circles are preserved under a Möbius transformation (p. 148). If we take circles A and B in Figures 1 and 3 to be instances of the family C_2 in [29], p. 163, the line segment L is an instance of the family C_1. All members of C_1 intersect at ξ_- and ξ_+, which are encircled by members of C_2. All members of C_1 are orthogonal to all members of C_2. Since F sends ξ_- and ξ_+ to 0 and ∞, the C_1 circles under F must intersect at these points. At 0 and ∞, they conserve the C_1 angles at ξ_- and ξ_+. Hence, the C_1 circles under F must form rays emanating from 0 and pointing radially to infinity. F(L) can be regarded as two rays emanating from 0 and pointing in opposite directions to infinity. Since A and B are orthogonal to L at ξ_- and ξ_+, the circles F(A) and F(B) must be orthogonal to F(L) at all intersections. But that is not enough to establish that F(A) and F(B) are concentric. We only know that they are because all members of C_1 under F emanate as rays from 0. Since A and B are orthogonal to all members of C_1, F(A) and F(B) are orthogonal to all of the rays, so they are concentric. Hence, we affirm the proposition that *Any two non-intersecting, non-concentric circles can be mapped to concentric circles by means of a suitable Möbius transformation*.