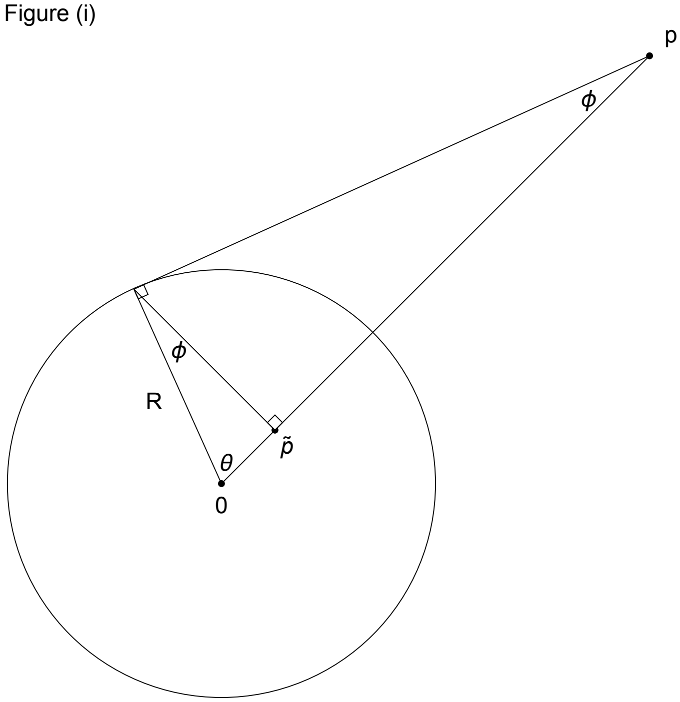
Needham, Ch. 3, Ex. 1, p. 181.

G. Palmer, March 8, 2021, 8 am PDST

1 In each of the figures below, show that p and $\tilde{p}$ are symmetric with respect to the circle. The dashed lines are not strictly part of the constructions, rather they are intended to be helpful or suggestive.

Figure (i)

Out[•]=

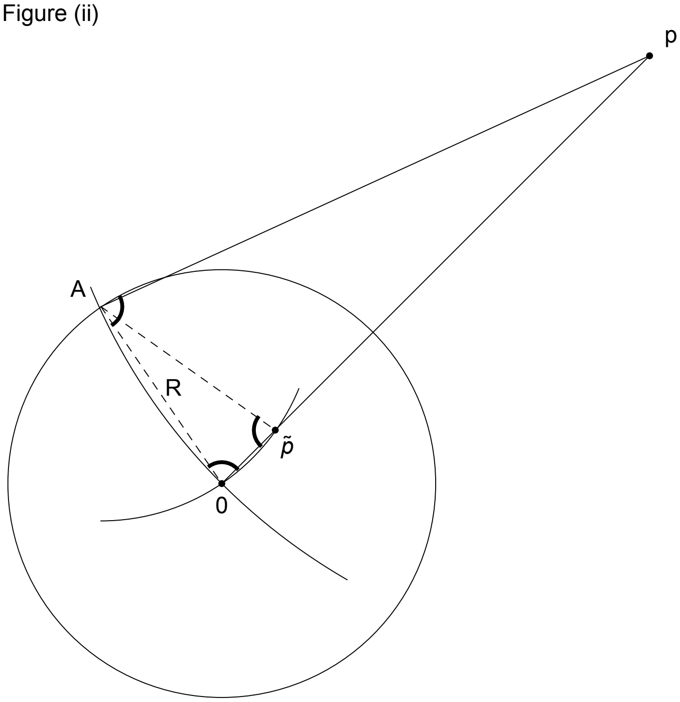


1 In each of the figures below, show that p and $\tilde{p}$ are symmetric with respect to the circle. The dashed lines are not strictly part of the constructions, rather they are intended to be helpful or suggestive.

(i) If the points p and $\tilde{p}$ are symmetric with respect to the circle, then they satisfy the equation (4) $\mathcal{I}_K(z)$, which requires that $\tilde{p}$ is the point in the same direction as p with reciprocal length. The centre q can be moved to the origin with no loss of generality. In Figure (i), since $\tilde{p}$ is on the ray p, they have the same direction. By analogy to Figure [1b] (reversing the tilde), we need only show that $|p| = R^2 / |\tilde{p}|$. By construction, the enclosing triangle and the small enclosed triangle share two interior angles (θ and $\pi/2$) and the side R, so they are similar and must also share the third angle ϕ. By dividing analogous sides (opposite ϕ and $\pi/2$) of similar triangles, $|p| / R = R / |\tilde{p}|$.

Figure (ii)

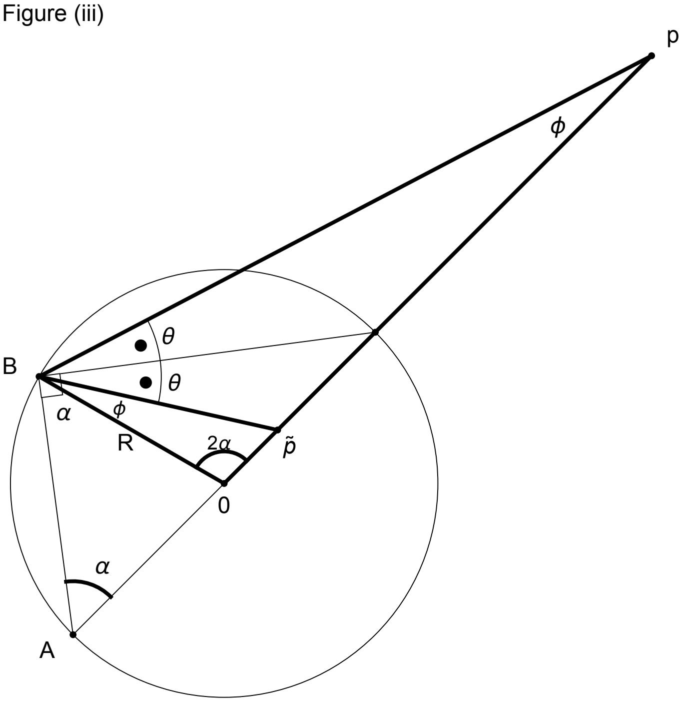
Out[•]=



(ii) As in (i), we let q = 0. Triangles 0Ap and 0A$\tilde{p}$ are isosceles by construction. Points p and $\tilde{p}$ still point in the same direction. The bold arcs show that the triangles have common base angles, so they are similar and we can write $| \ p \ | \ / R = R / \ | \ \tilde{p} \ |$, as in (i).

Figure (iii)

Out[•]=



(iii) We want to show that triangle B0p is similar to the enclosed triangle B0$\tilde{p}$. By construction, the two triangles share the angle 2α and the side marked R. Now we need only show that the angle at p equals ϕ at B. We again let q = 0.

First find ϕ using the right angle at B.

$\pi/2 = \theta + \phi + \alpha$	(sum of constituent angles in right angle)	(1)
$\phi = \pi/2 - \theta - \alpha$	(rearrange)	

Now write an equation using the enclosing triangle B0p. We use the fact that A0B is isosceles with two sides of length R and the fact that the angle of the ray B with p at the origin is twice the angle of the vector B–A with p–A at A.

$\pi = (2\theta+\phi)+\phi+2\alpha$	(sum of three interior angles)	(2)
$\phi = \pi/2 - \theta - \alpha$	(rearrange and divide by 2)	

The two angles labeled ϕ are in fact equal and the two triangles are similar, so we can write $|\tilde{p}|/R = R/|p|$ as in (i) and (ii), proving that collinear p and $\tilde{p}$ are symmetric with respect to the circle.