Needham, Chapter 2, Notes
G. Palmer, March 4, 2021, 2:10 pm PDST

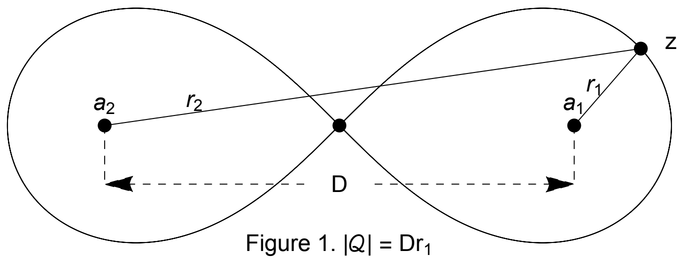


Figure 1. $|Q| = Dr_1$

P. 62, ¶ 2 Exercise

Indeed, since the ratio $\left[Q(z)/z^2\right]$ is easily seen [exercise] to tend to unity as |z| tends to infinity, we may say that Q(z) is ultimately equal to z^2 in this limit.

$$Q(z) = (z-a_1)\,(z-a_2)$$

$$\lim_{|z|\to\infty} Q(z) = \lim_{|z|\to\infty} (z-a_1)(z-a_2)/z^2 = \lim_{|z|\to\infty} \left(z^2 - a_1 - a_2\, z + a_1\, a_2\right)/z^2$$

$$= \lim_{|z|\to\infty} (1 - (a_1 + a_2)/z + a_1\, a_2/z^2) = 1 - 0 + 0 = 1$$

P. 62, ¶ 3 Exercise

Writing $D = |a_1 - a_2|$ for the distance between the foci, we see [exercise] that |Q| is ultimately equal to Dr_1 as z tends to a_1.

As z tends to a_1, r_2 tends to D (Figure 1). Then

$$|Q| = |(z-a_1)\,(z-a_2)| = |z-a_1|\,|z-a_2| = r_1\, D = Dr_1$$

P. 65 (4)

Needham presents (4) as an expected derivation from the previous equation. It is, but it is not totally obvious, so I am doing it as an exercise. From the first equation we have

$$\frac{1}{a-x} = \frac{1}{(a-k)} \frac{1}{\left[1-\left(\frac{x}{a-k}\right)\right]}$$

From (3), p. 64, we know that

$$\frac{1}{1-x} = \sum_{j=0}^{\infty} x^j$$

Now looking at the first equation, we see that the second term on the RHS has the same form as $\frac{1}{1-x}$ with $\frac{x}{a-k}$ substituted for x. Hence,

$$\frac{1}{a-x} = \frac{1}{(a-k)} \frac{1}{\left[1-\left(\frac{x}{a-k}\right)\right]} = \frac{1}{(a-k)} \sum_{j=0}^{\infty} \left(\frac{x}{a-k}\right)^j = \sum_{j=0}^{\infty} \frac{x^j}{(a-k)^{j+1}}, \text{ if and only if } |x| < |a-k|$$

With this equation, one can find the power series centred on any real number k with a singularity at a, and more besides, as the author shows by applying it to finding the expansion of $1/(1-x^2)$.

P. 75, ¶ 3 Exercise, R = e^3

By the root test, R = $\lim_{n\to\infty} \frac{1}{\sqrt[n]{|c_n|}}$.

The real function e^x may be written $e^x = \lim_{n\to\infty}(1 + x/n)^n$.

Apply root test to series P(z) = $\sum_{j=1}^{\infty}\left(\frac{j-3}{j}\right)^{j^2} z^j$.

$$c_j = \left(\frac{j-3}{j}\right)^{j^2} = \left(1 - \frac{3}{j}\right)^{j^2}$$

$$|c_j| = \left|\left(1 - \frac{3}{j}\right)^{j^2}\right|$$

$$R = \lim_{j\to\infty} \frac{1}{\sqrt[j]{|c_j|}} = \lim_{j\to\infty} \frac{1}{\sqrt[j]{\left|\left(1-\frac{3}{j}\right)^{j^2}\right|}} = \lim_{j\to\infty} \frac{1}{\left|\left(1-\frac{3}{j}\right)^{j^2}\right|^{\frac{1}{j}}} = \lim_{j\to\infty} \frac{1}{\left|1-\frac{3}{j}\right|^{j}}$$

$$= \frac{1}{e^{-3}} = e^3$$

P. 77 [exercise], Derive (18) from (17).

$$\frac{1}{1+z^2} = \sum_{j=0}^{\infty} \frac{1}{2i}\left[\frac{1}{(-i-k)^{j+1}} - \frac{1}{(i-k)^{j+1}}\right] Z^j \qquad (17)$$

$$\frac{1}{1+x^2} = \sum_{j=0}^{\infty} \frac{1}{2i}\left[\frac{1}{(-i-k)^{j+1}} - \frac{1}{(i-k)^{j+1}}\right] X^j$$ with k confined to the real line

$$= \sum_{j=0}^{\infty} \frac{1}{2i}\left[\frac{1}{\left(\sqrt{1+k^2}\, e^{-i\phi}\right)^{j+1}} - \frac{1}{\left(\sqrt{1+k^2}\, e^{i\phi}\right)^{j+1}}\right] X^j$$ (because Arg[-i-k] = -Arg[i-k]) = -ϕ)

$$= \sum_{j=0}^{\infty} \frac{1}{2i}\left[\frac{(e^{i\phi})^{j+1}}{\sqrt{1+k^2}^{\,j+1}(e^{-i\phi})^{j+1}(e^{i\phi})^{j+1}} - \frac{(e^{-i\phi})^{j+1}}{\sqrt{1+k^2}^{\,j+1}(e^{-i\phi})^{j+1}(e^{i\phi})^{j+1}}\right] X^j$$

$$= \sum_{j=0}^{\infty} \frac{1}{2i}\left[\frac{e^{i(j+1)\phi} - e^{-i(j+1)\phi}}{\sqrt{1+k^2}^{\,j+1}}\right] X^j$$

$$= \sum_{j=0}^{\infty} \frac{1}{2i}\left[\frac{2i\sin[(j+1)\phi]}{\sqrt{1+k^2}^{\,j+1}}\right] X^j$$

$$\frac{1}{1+x^2} = \sum_{j=0}^{\infty}\left[\frac{\sin[(j+1)\phi]}{\sqrt{1+k^2}^{\,j+1}}\right] X^j$$

P. 80 *"Here we have left it to you to show that the general term in the penultimate line is indeed* $(a+b)^n / n!$*."*

For a proof by induction, the first four terms of the series found by multiplying out are,

$$(a+b)^0 + (a+b)^1 + \frac{(a+b)^2}{2!} + \frac{(a+b)^3}{3!} + \ldots = 1 + (a+b) + \left[\frac{a^2+2ab+b^2}{2!}\right] + \left[\frac{a^3+3a^2b+3ab^2+b^3}{3!}\right] +$$

These are the values shown on p. 80, line 2, so either $(a+b)^0$ or $(a+b)^1$ may be taken as the the base case. We can see that each of the three terms can be written $\frac{(a+b)^n}{n!}$. Set n to be an arbitrary positive integer k and assume that the result is true, so that

$$\frac{(a+b)^k}{k!}$$

is a valid member of the series. For each successive term, the numerator is multiplied by (a+b) and the factorial n! = k! in the denominator is multiplied by (n+1) = (k+1).

$$\frac{(a+b)\,(a+b)^k}{(k+1)\,k!} = \frac{(a+b)^{k+1}}{(k+1)\,!}$$

This still has the structure $\frac{(a+b)^n}{n!}$, so we conclude that the general term is $\frac{(a+b)^n}{n!}$.

P. 89 suggested exercises

How does the shape of the ellipse change as we vary c?

As c increases, the ellipse approaches a circle. As c decreases, the ellipse approaches a line segment from −1 to 1.

How do we recover the real cosine function [from the ellipse] as c tends to zero?

I will give two answers. Some might find (a) to be insufficient.

(a) Let $z = t - \mathrm{i}c$ as in (26). Then

As c $\longrightarrow$ 0, $\cos(t - \mathrm{i}c) \to \cos(t)$. Since t is real, we have recovered the real cosine function.

(b) We can expand cos z. Let z = t-ic, as in (26). Write $\cos z = (1/2)\, e^{\mathrm{i}z} + e^{-\mathrm{i}z}$. Then

$$\cos z = (1/2)\left(e^{(c+\mathrm{i}t)} + e^{-(c+\mathrm{i}t)}\right),$$

As c $\longrightarrow$ 0,

$$\cos z = (1/2)\left(e^{\mathrm{i}t} + e^{-\mathrm{i}t}\right) = \cos t$$

We have again recovered the real cosine function. Vasco (p.c., Forum) has shown that one can also recover the real cosine function as t goes to zero.

What is the orbit of cos z as z travels eastward along the line y = c, above the real axis?

The orbit is an ellipse on which w travels in the clockwise direction.

What is the image of the vertical line x = c under z $\longrightarrow$ cosh z?

We can describe the line as c + it.

$$\cosh(c + \mathrm{i}t) = \cos(-t + \mathrm{i}c) = \cos(-z)$$

The image is the anticlockwise ellipse in (26).

What is the orbit of sin z as z travels eastward along the line y = c; how does it differ from the orbit of cos z; and is the resulting variation of |sin z| consistent with the modular surface shown in [25].

We can write the traversal of z as t + ic and plot sin(t+ic) from $t = -\pi$ to π. The orbit is a an ellipse of identical shape (given the same value of c) to that in (26), but the traversal of w begins and ends at $-\pi/2$ instead of $-\pi$ and the direction of travel is clockwise rather than anticlockwise. See below.

and is the resulting variation of |sin z| consistent with the modular surface shown in [25].

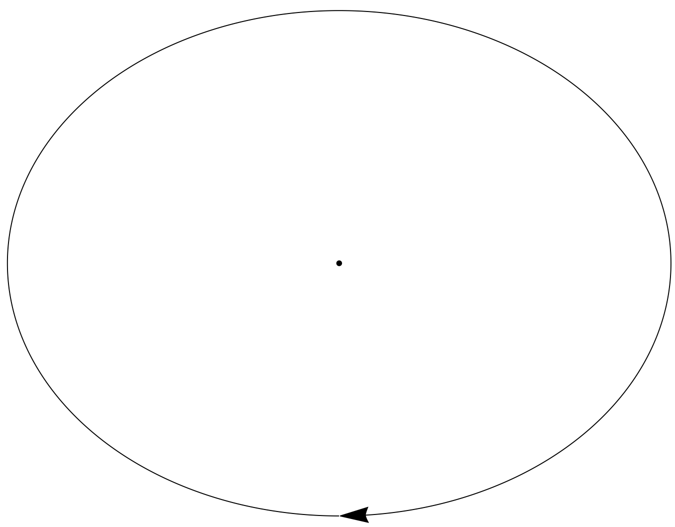
We are restricting z = t + ic to the value c, which is comparable to the y value in [25]. Then at c = 0 (cf. y = 0), the curve of the modular surface becomes sin t, which is analogous to sin x, as in [25]. It is consistent.

```
Clear["Global`*"]
```

```
v[z_] := {Re @ z, Im @ z};
```

```
zPts = I + Range[-Pi, Pi, .01];
(* t + ic, i = 1 *)
```

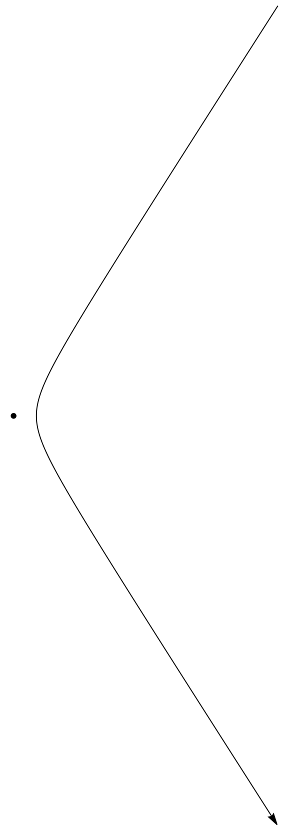
```
Graphics[{ Arrow @ v @ Sin[zPts], Point @ v @ 0}]
```



Before reading on, try using the ideas in [26] to sketch for yourself the image under z ⟶ cos z of a vertical line. The result is a hyperbola. See below.

```
zPts = 1 + I Range[-Pi, Pi, .01]; (* c + it, c = 1 *)
```

```
Graphics[{ Arrow @ v @ Cos @ zPts, Point @ v @ 0}]
```



P. 94 Figure [32] comparison

Out[•]=

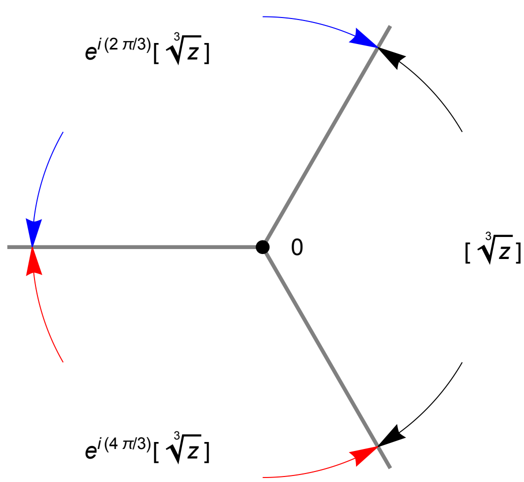


Figure 2. Multifunction $\sqrt[3]{z}$ on unit circle
as z approaches π in positive or negative direction
cf. Needham [32], p. 94

P. 98 *we define log(z) to be any complex number such that* $e^{\log(z)} = z$. *If follows [exercise] that*

log(z) = ln|z| + i arg(z).

Let z = |z| $e^{i \arg(z)}$. Then

$$\log(z) = \log|z| + \log\left(e^{i \arg(z)}\right) = \log|z| + i \arg(z)$$

P. 98, 99. *Check that you understand (roughly) the shapes of the illustrated image paths.*

Out[•]=

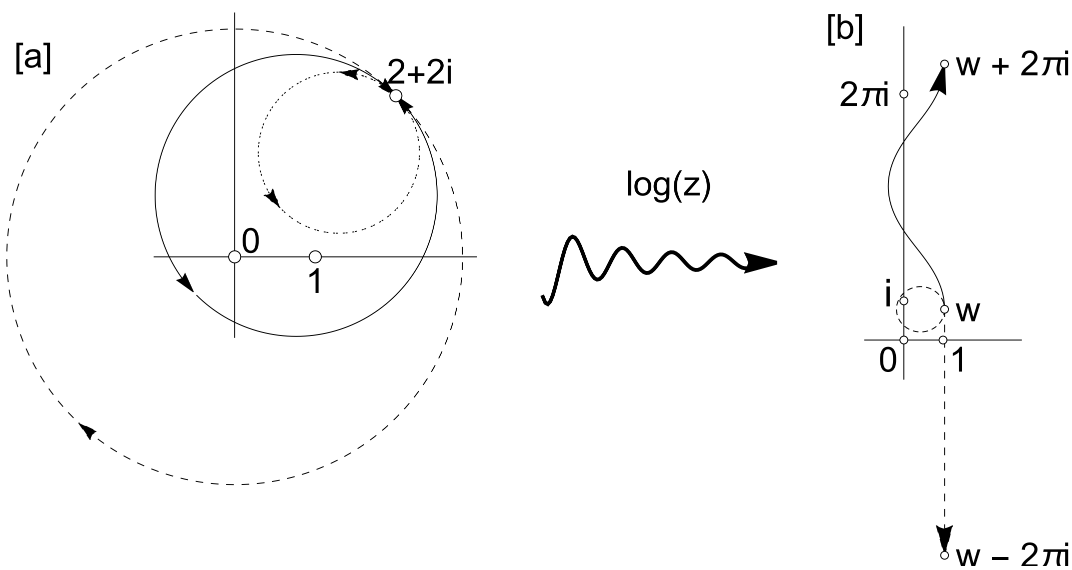


Figure 3. Facsimile [36], p. 99

All three circles in Figure 3 were plotted with angles from $\pi/4$ to $9\pi/4$. That is, they started and ended at the point 2+2i. Notice that only the outer dashed circle has a clockwise direction of traversal. This was implemented by reversing the order of the list of points on the circle.

The small dotted circle in Figure 3b was plotted directly with Mathematica Log[] applied to the points of the dotted circle in 3a. Log[] apparently converts angles to principal values.

The solid circle in 3a, which encircles 0, requires a log function that preserves angles outside those of the principal values. The pseudocode is

log(z) = **If** $-\pi < \mathrm{Arg}[z] < \pi/4$

then ln|z| + i (2π + Arg(z))

else ln|z| + i Arg(z)

Since |z| decreases to the point where Arg(z) = $5\pi/4$ and then increases as z approaches the starting point, the curve appears to bulge leftward.

The dashed circle in 3a has constant |z|. The corresponding arrow in 3b reflects the fact that log $re^{i\theta} = \ln(r) + i\theta$. Hence, we see a straight arrow at ln $(2\sqrt{2}) \simeq 1.04$ pointing south to reflect the clockwise direction of traversal of z.

P. 99-100. [Exercise] *Another problem with restricting ourselves to the principal branch is that the familiar rules for the logarithm break down. For example, Log(ab) is not always equal to Log(a) + Log(b); try a = -1 and b = 1, for example. However, if we keep all values of the logarithm in play then it is true [exercise] that*

log(ab) = log(a) + log(b) and log(a/b) = log(a) – log(b),

in the sense that every value of the LHS is contained amongst the values of the RHS, and vice versa.

To *keep all values of the logarithm in play* is to write the logarithm so that every branch is represented. In the equation Log(z) = ln|z| + iArg(z), capital letters indicate principal values of arg and log. Let n be any member of an infinite series 0, ±1, ±2, ±3, ... Then one can write arg(z) = Arg(z) + 2nπi. Then every possible branch is represented in an expression such as

$$\log(z) = \text{Log}(z) + n2\pi i \qquad (1)$$

or

$$\log(z) = \ln|z| + i\text{Arg}(z) + n2\pi i \qquad (2)$$

or

$$\log(z) = \ln|z| + i\arg(z) \qquad (3) \quad \text{arg() rather than Arg()}$$

Each of these contains the principal values of Log and Arg, and the values obtained by adding integer multiples of 2πi.

To add the logarithms of two complex numbers, as one would do to calculate log(ab), one must allow for the possibility that a and b have angles that sum to an angle beyond the principal values. One also requires a notation to show that they may have different angles. We introduce a new letter m, which has the same range of values as n, but can be different for any two particular instances of a and b. Then, we can keep all possible values in play by writing:

$$a = r_a e^{i\theta}, \qquad b = r_b e^{i\phi},$$

where r_a = |a|, r_b = |b|, θ = arg(a) and ϕ = arg(b). We introduce two new integers n and k, defined in the same way as m.

We apply this to log(ab) and log(a/b)

$$\log(ab) = \log(r_a e^{i\theta}\, r_b e^{i\phi})$$

$$= \log(r_a r_b e^{i\theta} e^{i\phi})$$

$$= \ln(r_a r_b) + i\arg(e^{i(\theta+\phi)}) \text{ by log of product of exponents}$$

$$= \ln(r_a r_b) + i(\theta+\phi+2k\pi)$$

$$= \ln r_a + \ln r_b + i(\theta+2m\pi) + i(\phi+2n\pi), \text{ where } k = m + n$$

$$= [\ln r_a + i(\theta+2m\pi)] + [\ln r_b + i(\phi+2n\pi)]$$

$$= \log(a) + \log(b) \quad \text{QED}$$

$$\log(a/b) = \log(r_a e^{i\theta} / r_b e^{i\phi})$$

$$= \ln(r_a/r_b) + i\arg(e^{i\theta}/e^{i\phi})$$

$$= \ln(r_a/r_b) + i\arg(e^{i(\theta-\phi)}) \qquad \text{by quotient of exponents}$$

$$= \ln(r_a/r_b) + i(\theta - \phi + 2l\pi) \qquad \text{by def of arg}$$

$= \ln (r_a/r_b) + i(\theta - \phi + 2(m\text{-}n)\pi)$ where k = m – n

$= \ln r_a - \ln r_b + i(\theta + 2m\pi) - i(\phi + 2n\pi)$

$= [\ \ln r_a + i(\theta + 2m\pi)\ ] - [\ \ln r_b + i(\phi + 2n\pi)\ ]$, by rearranging

$=$ log(a) – log(b)   QED

Apply this to Needham's example of a = -1 and b = i, where

$-\pi i/2 = \text{Log}(ab) \neq \text{Log}(a) + \text{Log}(b) = \pi i + \pi i/2 = 3\pi i/2.$

But keeping all the values of the logarithm in play by using log instead of Log,

$\log(ab) = \log(-i) = \ln(1) + i\ (\pi + \pi/2 + 2k\pi) = i\ (3\pi/2 + 2k\pi)$

$\log(a) = \log(-1) = \ln(1) + i(\pi + 2m\pi) = i\ (\pi + 2m\pi)$

$\log(b) = \log(i) = \ln(1) + i\ (\pi/2 + 2n\pi) = i\ (\pi/2 + 2n\pi)$

$\log(a) + \log(b) = i\ (\pi + 2m\pi) + i\ (\pi/2 + 2n\pi) = i\ (3\pi/2 + 2j\pi)$, where j = m + n

This shows that while Log[-1 × i] is not always equal to Log(-1) + Log(i), by keeping all the values of the multifunction in play, we can write log(-1 × i) = log(-1) + log(i), because j = (m + n) has the same range of values as k.

P. 101. *Next, consider the three branches of* $z^{1/3}$. *Recalling that the principal branch* $\left[z^{\frac{1}{3}}\right]$ *of this function is* $\sqrt[3]{r}\ e^{i(\theta/3)}$, *where* θ *again represents the principal angle, you can easily check that* $e^{\frac{1}{3}\text{Log}(z)} = \left[z^{\frac{1}{3}}\right]$.

$$e^{\frac{1}{3}\text{Log}(z)} = e^{\frac{1}{3}(\ln r + i\theta)} = e^{\frac{1}{3}\ln r}\ e^{\frac{1}{3}i\theta} = r^{\frac{1}{3}} e^{\frac{1}{3}i\theta} = \left[z^{\frac{1}{3}}\right]$$

We can write $\left[z^{\frac{1}{3}}\right]$ because Log(z) is confined to the principal values.

P. 102, [exercise]. Using

$z^k = e^{k\log(z)}$ (30)

Let $z = re^{i\theta}$

show that $\left[z^{(a+ib)}\right] \equiv r^a\ e^{-b\theta}\ e^{i(a\theta + b\ln r)}$

From (30), using only the principal values of log(z),

$$\left[z^{(a+ib)}\right] \equiv e^{(a+ib)\,\text{Log}(z)} = e^{(a+ib)\,(\ln|z| + i\,\text{Arg}(z))}$$

$$= e^{(a\ln(r) + ai\,\text{Arg}(z) + ib\ln(r) - b\text{Arg}(z))}$$

$$= e^{a\ln(r)}\ e^{i\,(a\,\text{Arg}(z) + b\ln(r))}\ e^{-b\text{Arg}(z)}$$

$$= r^a\ e^{-b\theta}\ e^{i(a\theta + b\ln(r))}$$

This is what we wanted.

P. 102. If z travels an closed loop circling the origin n times:

$$z^{(a+ib)} = e^{(a+ib)(\ln|z| + i\,\mathrm{Arg}(z) + 2\,in\pi)}$$

$$= e^{[(a\,r + ai\,\mathrm{Arg}(z) + ib\ln(r) - b\mathrm{Arg}(z)) + (a2in\pi - b2n\pi)]}$$

$$= r^a\, e^{-b\theta}\, e^{i(a\theta + b\ln(r))}\, e^{(ai2n\pi - b2n\pi)} \qquad \text{subst. result of previous ex.}$$

$$= r^a\, e^{-b\theta}\, e^{i(a\theta + b\ln(r))}\, e^{i2\pi ma}\, e^{-2\pi mb}$$

$$= e^{i2\pi n a}\, e^{-2\pi n b}\left[z^{a+ib}\right] \qquad \text{subst. frm prev. ex. and rarrange}$$

This is what we wanted.

P. 110. *For example, the exact version of [44] is* $< e^z >_C = e^0 = 1$*, and this implies [exercise] that* $\int_0^{2\pi} e^{\cos\theta} \cos(\sin(\theta))\, d\theta = 2\pi$.

Since the unit circle (p.109) in [44a] is centred at the origin, k = 0 and R = 1. With these substitutions, (1) can be rewritten as (2)

$$< f(z) >_C = \frac{1}{2\pi} \int_0^{2\pi} f\left(k + \mathrm{Re}^{i\theta}\right) d\theta = f(k) \qquad (1)$$

$$< e^z >_C = 1 = \frac{1}{2\pi} \int_0^{2\pi} e^{0 + e^{i\theta}}\, d\theta \qquad (2)$$

Multiply through by 2π and apply Euler's equation recursively.

$$2\pi = \int_0^{2\pi} e^{\cos\theta + i\sin(\theta)}\, d\theta$$

$$= \int_0^{2\pi} e^{\cos\theta}\, e^{i\sin(\theta)}\, d\theta$$

$$= \int_0^{2\pi} e^{\cos\theta} (\cos(\sin(\theta) + i\sin(\sin\theta))\, d\theta$$

$$= \int_0^{2\pi} e^{\cos\theta} \cos(\sin(\theta))\, d\theta$$

This is what we wanted to show.