Needham, Chapter 2, Exercise 9
G. Palmer, July 14, 2020, 10 pm PST

9. Here is an attempt [ultimately doomed] at using real methods to expand $H(x) = 1 / (1 + x^2)$ *into a power series, centred at* $x = k$*, i.e., into a series of the form* $H(x) = \sum_{j=0}^{\infty} c_j X^j$*, where* $X = (x\text{-}k)$*. According to Taylor's Theorem,* $c_j = H^{(j)}(k) / j!$*, where* $H^{(j)}(k)$ *is the* j^{th} *derivative of* H*.*

Vasco has pointed out to me in personal communication (Forum) that unlike $1 / (1 - x^2)$, $1 / (1 + x^2)$ cannot be factorized, so one cannot use factors to decompose the result into partial fractions. Hence, the resort to Taylor's Theorem.

(i) Show that $c_0 = 1 / (1 + k^2)$ *and* $c_1 = -2k / (1 + k^2)^2$*, and find* c_2*. Note how it becomes increasingly difficult to calculate the successive derivatives.*

$$c_0 = H^{(0)}(k) / 0!$$

$$= H^{(0)}(x) = 1 / (1 + x^2)$$

$$= 1 / (1 + k^2) \text{ at (k)} \qquad (1)$$

$$c_1 = H^{(1)}(k) / 1!$$

$$= H^1(k)$$

$$= \frac{d}{dx}(1 + x^2)^{-1} \text{ at (k)}$$

$$= \frac{d}{dx}u^{-1}, \text{u} = 1 + x^2$$

$$= -u^{-2} \frac{du}{dx}$$

$$= -(1 + x^2)^{-2} \text{ 2x}$$

$$= -2x / (1 + x^2)^2$$

$$= -2k / (1 + k^2)^2 \qquad (2)$$

Since H(x) = 1/(1 + x) then

$$\text{H(x)}(1 + x^2) = 1 \qquad (3)$$

Differentiate (3) wrt x:

$$H^{(1)}(x)\left(1+x^2\right)+H(x)(2\mathrm{x})=0 \qquad (4)$$

$$H^{(1)}(x) \;=\; -2\,\mathrm{xH}(x)\Big/\left(1+x^2\right)$$

$$= -2\ \mathrm{x}\Big/\left(1\ +\ x^2\right)^2$$

$$c_1=-2\,k\Big/\left(1\ +\ k^2\right)^2$$

Differentiate (4) wrt x:

$$H^{(2)}(x)\left(1+x^2\right)+H^{(1)}(x)(2\mathrm{x})+H^{(1)}(x)\,(2\,x)\ +\ 2\,H(x)=0$$

Rearranging

$$H^{(2)}(x)\left(1+x^2\right)= -(\ H^{(1)}(x)(2\mathrm{x})+H^{(1)}(x)\,(2\,x)\ +\ 2\,H(x)\ )$$

$$H^{(2)}(x)\ =\ -(\ H^{(1)}(x)(2\mathrm{x})+H^{(1)}(x)\,(2\,x)\ +\ 2\,H(x)\ \Big)\left(1+x^2\right)^{-1}$$

$$= -[(4H^{(1)}(x)\mathrm{x}\ \ +2\Big/\left(1+x^2\right)]\left(1+x^2\right)^{-1}$$

$$= [-4\mathrm{x}\ (-2\ \mathrm{x}\Big/\left(1\ +\ x^2\right)^2) - 2\Big/\left(1+x^2\right)]\left(1+x^2\right)^{-1}$$

$$= \left[8\,x^2\,/\left(1\ +\ x^2\right)^2 - 2\Big/\left(1\ +\ x^2\right)\right]\left(1\Big/\left(1+x^2\right)\right)$$

$$= 8\,x^2\,/\left(1\ +\ x^2\right)^3 - 2\Big/\left(1\ +\ x^2\right)^2$$

$$= 8\,x^2\,/\left(1\ +\ x^2\right)^3 - 2\left(1+x^2\right)\Big/\left(1\ +\ x^2\right)^3$$

$$= \left(8\,x^2-2-2\,x^2\right)/\left(1\ +\ x^2\right)^3$$

$$= \left(6\,x^2-2\right)/\ \left(1\ +\ x^2\right)^3$$

So

$$c_2=H^{(2)}(k)\Big/2\,!$$

$$= \left(6\,x^2 - 2\right) / \left(1 + x^2\right)^3 / 2$$

$$= \left(3\,x^2 - 1\right) / \left(1 + x^2\right)^3$$

$$= \left(3\,k^2 - 1\right) / \left(1 + k^2\right)^3$$

We check it in *Mathematica*.

```
Clear["Global`*"];
x = k;
c0 = 1 / (1 + x^2);
c1 = D[c0, x];
c2 = (D[c1, x] / 2) // Simplify; Print["c2 = ", c2, ",    c1 = ", c1, "    c0 = ", c0];
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$$c_2 = \frac{3\,k^2 - 1}{(k^2 + 1)^3}, \quad c_1 = -\frac{2\,k}{(k^2 + 1)^2} \quad c_0 = \frac{1}{k^2 + 1}$$

(ii) Recall or prove that the nth derivative of a product AB of two functions A(x) and B(x) is given by Leibniz's rule

$$(AB)^{(n)} = \sum_{j=0}^{n} \binom{n}{j} A^{(j)}\, B^{(n-j)} \qquad \text{("r" in original binomial replaced by "j")}$$

I will try a proof by induction, adapting a proof presented in Wikipedia. We can choose the (0) derivative as the base case, which is just the function itself. The base case is true by definition, but it is also given by Leibnitz's formula.

$$(\mathrm{AB})^{(0)} = \sum_{j=0}^{0} \binom{0}{0} A^{(0)}\, B^{(0-0)}$$

$$\mathrm{AB} = \frac{1}{1} \cdot \mathrm{AB}$$

We could also choose the first derivative as the base case, as it is the commonly used form of Leibnitz's product rule.

$$(\mathrm{AB})' = A'\,B + A\,B'$$

Find the (k) and (k+1) derivatives, where k is an arbitrary integer. Assume that the formula is true for the (k) derivative.

$$(\mathrm{AB})^{(k)} = \sum_{j=0}^{k} \binom{k}{j} A^{(j)}\, B^{(k-j)} \qquad (1)$$

$$(\mathrm{AB})^{(k+1)} = \left((\mathrm{AB})^{(k)}\right)^{(1)} = \left[\sum_{j=0}^{k} \binom{k}{j} A^{(j)} B^{(k-j)}\right]'$$

$$= \sum_{j=0}^{k} \binom{k}{j} A^{(j+1)} B^{(k-j)} + \sum_{j=0}^{k} \binom{k}{j} A^{(j)} B^{(k+1-j)}$$ (2) Product rule.

$$= \sum_{j=1}^{k+1} \binom{k}{j-1} A^{(j)} B^{(k+1-j)} + \sum_{j=0}^{k} \binom{k}{j} A^{(j)} B^{(k+1-j)}$$ Change range of sum in first term.

$$= \binom{k}{k} A^{(k+1)} B + \sum_{j=1}^{k} \binom{k}{j-1} A^{(j)} B^{(k+1-j)} + \sum_{j=1}^{k} \binom{k}{j} A^{(j)} B^{(k+1-j)} + \binom{k}{0} \mathrm{AB}^{(k+1)}$$

Extract last and first summands.

$$= A^{(k+1)} B + \sum_{j=1}^{n} \left(\binom{k}{j-1} + \binom{k}{j}\right) \mathtt{A}^{(\mathtt{j})}\ \mathtt{B}^{(\mathtt{k+1-j})} + \mathrm{AB}^{(k+1)}$$

$$= A^{(k+1)} B + \sum_{j=1}^{n} \binom{k+1}{j} \mathtt{A}^{(\mathtt{j})}\ \mathtt{B}^{(\mathtt{k+1-j})} + \mathrm{AB}^{(k+1)}$$ Pascal's rule.

$$= \sum_{j=0}^{k+1} \binom{k+1}{j} A^{(j)} B^{(k+1-j)}$$

$$= (\mathrm{A}\, B)^{(k+1)}$$

Note as a point of interest that in (1), the sum of the derivative orders of A and B is (k), which is the derivative sought. In (2), and all subsequent equations the sum *for each term* of the derivative orders of A and B is (k+1), which is the derivative sought.

By applying this result to the product $\left(1 + x^2\right) H(x)$*, deduce that*

$$\left(1 + k^2\right) H^{(n)}(k) + 2nkH^{(n-1)}(k) + n(n-1)H^{(n-2)}(k) = 0.$$

Why are we applying the result to this product, which equals 1?

$$\mathrm{AB} = \left(1 + x^2\right) \mathrm{H} = 1 \quad (\text{because } \mathrm{H}(\mathrm{x}) = 1 \big/ \left(1 + x^2\right))$$

$$\left(\left(1 + x^2\right) H\right)^{(n)} = \sum_{j=0}^{n} \binom{n}{j} \left(1 + x^2\right)^j H^{(n-j)} = 1^{(n)} = 0 \ (\text{for } \mathrm{n} > 0)$$

Expand Leibnitz's formula into a series.

$$((1 + x^2) H)^{(n)} = n!/(n!\,0!)\,(1 + x^2)^{(0)} H^{(n-0)}(x) + (n!/(n-1)!\,1!)\,(1 + x^2)^{(1)} H^{(n-1)} + (n!/(n-2)!\,2!)\,(1 + x^2)^{(2)} H^{(n-2)} \ldots$$

$$= (1 + x^2) H^{(n)} + \text{n2x}\, H^{(n-1)} + (n(n-1)/2) \cdot 2\, H^{(n-2)} + (\text{n!/ j!(n-j)!}) \cdot 0 \cdot H^{(n-3)}$$

$$= (1 + x^2) H^{(n)} + 2\,\text{nxH}^{(n-1)} + n\,(n-1)\, H^{(n-2)} + 0 = 0$$

Substitute k for x, including the x in H = H(x).

$$(1 + k^2) H^{(n)}(k) + 2\text{nk}\, H^{(n-1)}(\text{k}) + n\,(n-1)\, H^{(n-2)}(\text{k}) = 0$$

This is what we wanted, but why did we want it? It is an equation for the nth derivative of AB = $(1 + x^2)$ H. Apparently the reason we want it is to obtain an equation that relates the coefficients in terms of k. Since the coefficients are successive functions of $H^{(j)} / j!$, they fit a recurrence relation. But, they are merely the coefficients of X^n in the Taylor's series, not the terms of the series itself, which would be $(H^{(j)} / j!)\, X^j$. In this problem, instead of searching for a recurrence relation for a series of integers, we search for a recurrence relation for a series of real constants, the coefficients of X^n. Then if we were to write an equation in the format $S_{n+2} = \text{pS}_{n+1} + \text{qS}_n$, the S_i would not refer to the terms of the Taylor's series, but rather to the coefficients of the Taylor's series, which we can only find with the Leibnitz expansion of $(\text{AB})^{(n)}$. For this reason, the method applied here is a bit more involved than that demonstrated in Exercise 30, p. 50.

(iii) Deduce from the previous part that the recurrence relation for the c_j's is:

$$(1 + k^2)\, c_n + 2\, k c_{n-1} + c_{n-2} = 0$$

which does have constant coefficients.

Why search for a recurrence relation? Vasco pointed out that if we can find the first few terms of the series and a recurrence relation for them, then we can find all the terms. Given the difficulty of calculating higher derivatives, it is more efficient to find a recurrence relation.

From (i): $c_0 = H^{(0)}(k) / 0!, \quad c_1 = H^{(1)}(k) / 1!, \quad c_2 = H^{(2)}(k) / 2!$

$$H^{(0)}(k) = c_0, \qquad H^{(1)}(\text{k}) = c_1, \qquad H^{(2)}(\text{k}) = 2\, c_2$$

Substitute each of these into the equation from (ii) using n = 2.

$$(1 + k^2)\, 2\, c_2 + 2\, \text{nkc}_1 + n\,(n-1)\, c_0 = 0$$

$$(1 + k^2)\, 2\, c_2 + 4\, \text{kc}_1 + 2\, c_0 = 0 \qquad \text{(substituting, n = 2)}$$

$$\left(1 + k^2\right) c_2 + 2\,\mathrm{kc}_1 + c_0 = 0 \qquad \text{(dividing through by 2)}$$

This works for the first three terms, but Vasco pointed out that a more general proof is required.

From (ii) use

$$\left(1 + k^2\right) H^{(n)}(k) + 2\mathrm{nk}\, H^{(n-1)}(\mathrm{k}) + n\,(n-1)\, H^{(n-2)}(\mathrm{k}) = 0$$

and $c_j = H^{(j)}(k) \big/ j\,!$, *where* $H^{(j)}(k)$ *is the* j^{th} *derivative of* H, but we can write $H^{(n)}(k) = \mathrm{n}!\, c_n$, so

$$\left(1 + k^2\right) n!\, c_n + 2\mathrm{n\,k\,(n{-}1)!}\, c_{n-1} + \mathrm{n(n\text{-}1)}(n-2)!\, c_{n-2} = 0$$

Divide through by (n)!

$$\left(1 + k^2\right) c_n + 2\mathrm{k}\, c_{n-1} + c_{n-2} = 0 \qquad \text{(because n (n-1)! = n!, and n(n-1)}(n-2)! = \text{n!)}$$

This is what we wanted.

(iv) Solve this recurrence relation, and hence recover the result (17) on p. 76.

The result we wish to recover is

$$\frac{1}{1+z^2} = \sum_{j=0}^{\infty} \frac{1}{2\,i}\left[\frac{1}{(-i-k)^{j+1}} - \frac{1}{(i-k)^{j+1}}\right] Z^j \qquad (17)$$

where Z = (z-k).

In Vasco's view (p.c., forum), to "solve" the recurrence relation means to find an expression for H(x) in terms of k and n so that one can calculate the coefficients of the power series in x." But at the present point in this exercise, it appears that we must first solve for the coefficients using the recurrence relation from (iii). And it seems we have to first seed the clouds with p^n, which we substitute for c_n. This is a bit of magic that I don't understand, but it results in a quadratic polynomial in p. Then I believe we can say that $c_n = p^n$ will solve this recurrence relation if $K_2\, p^2 + K_1\, p + 1 = 0$, where $K_2 = \left(1 + k^2\right)$ and $K_1 = 2\mathrm{k}$. Note that $p^0 = 1$.

$$\left(1 + k^2\right) p^2 + 2\,\mathrm{kp} + 1 = 0$$

Solve for p:

$$\mathrm{p} = \frac{-2\,k \pm \sqrt{4\,k^2 - 4\left(1+k^2\right)}}{2\left(1 + k^2\right)} = \frac{-k \pm i}{1 + k^2} = \frac{-k \pm i}{(-k-i)\,(-k+i)} = \frac{1}{-k-i} \text{ or } \frac{1}{-k+i}$$

The quadratic function has bumped us up to the complex plane, showing that this method fails to remain real. But we can persist in the complex plane and recover (17) using the complex numbers. Then if A and B are arbitrary complex numbers, $S_n = \frac{A}{(-k-i)^n} + \frac{B}{(-k+i)^n}$ (from Exercise 30, Part (ii), p. 50). But here we keep in mind that $S_n = c_n$.

$$c_0 = A + B = \frac{1}{1+k^2} \Longrightarrow B = \frac{1}{1+k^2} - A \qquad \text{because } (-k \pm i)^0 = 1$$

$$c_1 = \frac{A}{-k-i} + \frac{B}{-k+i} = \frac{-2k}{\left(1+k^2\right)^2}$$

With two equations in two unknowns, we can solve for A and B.

$$\frac{A}{-k-i} + \frac{\frac{1}{1+k^2} - A}{-k+i} = \frac{-2k}{\left(1+k^2\right)^2}$$

$$\frac{A(-k+i)}{1+k^2} + \frac{\left(\frac{1}{1+k^2} - A\right)(-k-i)}{1+k^2} = \frac{-2k}{\left(1+k^2\right)^2}$$

$$-Ak + Ai + \left(\frac{1}{1+k^2} - A\right)(-k-i) = \frac{-2k}{1+k^2} \qquad \text{multiply through by } \left(1+k^2\right)$$

$$-Ak + Ai - \frac{k}{1+k^2} - \frac{i}{1+k^2} + Ak + Ai = \frac{-2k}{1+k^2}$$

$$2Ai = \frac{k+i-2k}{1+k^2} = \frac{-k+i}{1+k^2}$$

$$A = \frac{1}{2i}\frac{-k+i}{1+k^2} = \frac{1}{2i(-k-i)}$$

$$B = \frac{1}{1+k^2} - \frac{1}{2i(-k-i)} = \frac{1}{1+k^2} - \frac{-k+i}{2i\left(1+k^2\right)} = \frac{2i+k-i}{2i\left(1+k^2\right)} = \frac{k+i}{2i\left(1+k^2\right)} = -\frac{1}{2i(-k+i)}$$

Substitute A and B into the expression for $S_n = c_n$.

$$c_n = \frac{\frac{1}{2i(-k-i)}}{(-k-i)^n} + \frac{-\frac{1}{2i(-k+i)}}{(-k+i)^n} = \frac{1}{2i}\left[\frac{1}{(-k-i)^{n+1}} - \frac{1}{(-k+i)^{n+1}}\right]$$

Since c_n is a complex number and $H(z) = \sum_{j=0}^{n} c_j Z^j$, where $Z = (z-k)$, we have recovered the expansion (17) on p. 76

$$\frac{1}{1+z^2} = \sum_{j=0}^{\infty} \frac{1}{2i}\left[\frac{1}{(-i-k)^{j+1}} - \frac{1}{(i-k)^{j+1}}\right] Z^j \qquad (17)$$

where Z = (z-k). If we want H(x) to be real, we must confine z and k to the real line. Or we might make the decision to obtain only a real solution earlier in the calculation, as in part (iii), Exercise 30, on p. 50. Since $\overline{A} = \frac{1}{-2i(-k+i)} = B$,

$$c_n = 2\,\mathrm{Re}\left[\frac{A}{2\,i\,(-k-i)^n}\right] = \mathrm{Re}\left[\frac{1}{(-k-i)^n}\right]$$

Then perhaps we can write

$$\frac{1}{1+x^2} = \sum_{j=0}^{\infty} \mathrm{Re}\left[\frac{1}{(-k-i)^j}\right] X^j$$