Needham, Chapter 2, Exercise 7
G. Palmer, June 26, 2020, 7:20 am PST

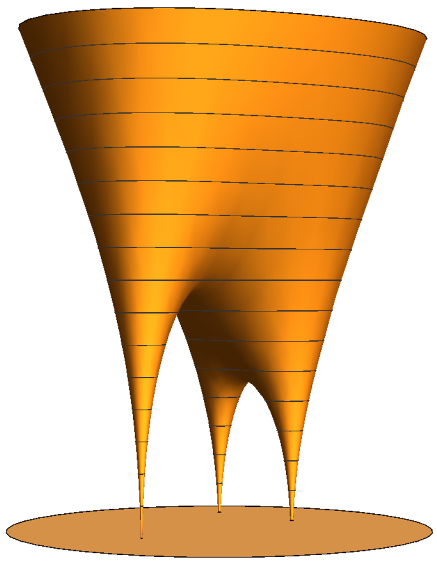

Figure 1. Modular surface |C(z)| of
C(z)=(z+1)(z−1)(z+1+i)

8. On page 62 we saw that the polar equation of the lemniscate with foci at ±1 is $r^2 = 2\cos 2\theta$. In fact James Bernoulli and his successors worked with a slightly different lemniscate having equation $r^2 = \cos 2\theta$. Let us call this the standard lemniscate.

(1) What are the foci of the standard lemniscate?

Wikipedia (https : // en.wikipedia.org / wiki / Lemniscate_of _Bernoulli) has

In polar coordinates $r^2 = 2\,a^2 \cos 2\theta$

where a and -a are the foci. From figure [9], the foci will be $\pm\sqrt{c}$, and c = 1. Then

$$r^2 = 2\left(\sqrt{c}\right)^2 \cos 2\theta = 2c \cos 2\theta = 2\cos 2\theta,$$ which is consistent with c = 1 on the unit circle.

For the standard lemniscate,

$$r^2 = \cos 2\theta \Rightarrow 2\,c = 1, \text{ so } c = 1/2.$$

Hence, the foci are $\pm\sqrt{c} = \pm\frac{1}{\sqrt{2}}$. We will let $a = \frac{1}{\sqrt{2}}$. Then the second focus is at $-a = -\frac{1}{\sqrt{2}}$.

(ii) What is the value of the product of the distances from the foci to a point on the standard lemniscate?

Since this value is constant, the value at the origin will serve. This value is

$$k^2 = |r_1 r_2| = |\text{a (-a)}| = |\frac{1}{\sqrt{2}}(-\frac{1}{\sqrt{2}}| = \frac{1}{2}$$

(iii) Show that the Cartesian equation of the standard lemniscate is $\left(x^2 + y^2\right)^2 = \left(x^2 - y^2\right)$.

$$r_1 = |z-a| = |(x-a) + \text{iy}|$$

$$r_1{}^2 = ((x-a) + \text{iy})\,((x-a)-\text{iy}) = x^2 - 2\,a\text{x} + a^2 - (\text{x}-a)\text{ iy} + (\text{x}-a)\text{ iy} + y^2\text{::(by z } \bar{z} = |z|^2)$$

$$= x^2 - 2a_1\text{x} + a_1^2 + y^2$$

$$= x^2 - 2a\text{x} + a^2 + y^2$$

$$r_2 = |z + a| = |(\text{x+a}) + \text{iy}|$$

$$r_2{}^2 = ((\text{x+a})+\text{iy})((\text{x+a})-\text{iy}) = x^2 + 2a\text{x} + a^2 - (\text{x+a})\text{ iy} + (\text{x+a})\text{ iy} + y^2$$

$$= x^2 + 2\text{ax} + a_2^2 + y^2$$

$$r_1{}^2 r_2{}^2 = (\,x^2 - 2a\text{x} + a^2 + y^2)(x^2 + 2\text{ax} + a^2 + y^2)$$

$$= x^4 + 2ax^3 + a^2x^2 + x^2\,y^2$$

$$- 2ax^3 - 4a^2x^2 - 2a^3\text{x} - 2a\text{x}y^2$$

$$+ a^2x^2 + 2a^3\text{x} + a^4 + a^2y^2$$

$$+ x^2y^2 \qquad + 2a\text{x}y^2 + a^2y^2 + y^4$$

$$= \left(x^4 + 2x^2\,y^2 + y^4\right) - 2a^2x^2 + a^4 + 2a^2y^2$$

$$= \left(x^2 + y^2\right)^2 - 2a^2\left(x^2 - y^2\right) + a^4$$

We know from part (ii) that $r_1 r_2 = \text{k} = \frac{1}{2}$, so $r_1{}^2 r_2{}^2 = k^2 = \frac{1}{4}$.

$$\left(x^2+y^2\right)^2 \ - 2a^2\left(x^2-y^2\right)+a^4=\frac{1}{4}$$

$$\left(x^2+y^2\right)^2 \ - 2\left(\frac{1}{\sqrt{2}}\right)^2\left(x^2-y^2\right)+\left(\frac{1}{\sqrt{2}}\right)^4=\frac{1}{4}$$

$$\left(x^2+y^2\right)^2 \ - \left(x^2-y^2\right)=\frac{1}{4}-\frac{1}{4} \qquad \text{(by simplifying and rearranging)}$$

$$\left(x^2+y^2\right)^2=\left(x^2-y^2\right)$$

This is what we wanted.

Wikipedia (https://en.wikipedia.org/wiki/Lemniscate_of_Bernoulli) has

Its Cartesian equation is (up to translation and rotation):

$$\left(x^2+y^2\right)^2=2\,a^2\left(x^2-y^2\right)$$

where 2a is the distance between the foci. For the standard lemniscate, this would be

$$\left(x^2+y^2\right)^2=2\left(\pm\frac{\sqrt{2}}{2}\right)^2\left(x^2-y^2\right)= \ \left(x^2+y^2\right)^2=\left(x^2-y^2\right)$$