Needham, Chapter 2, Exercise 7
G. Palmer, June 26, 2020, 9:20 am

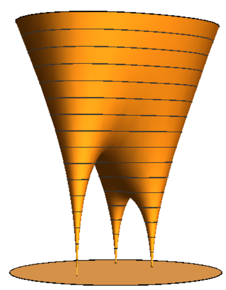

Figure 1. Modular surface |C(z)| of
C(z)=(z+1)(z−1)(z+1+i)

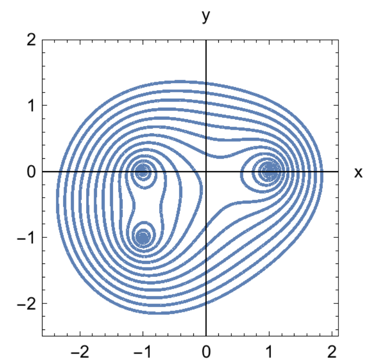

Figure 2. Cassinian curves |C(z)| = const. of
C(z)=(z+1)(z−1)(z+1+i)

7. Sketch the modular surface of C(z) = (z+1)(z-1)(z+1+i). Hence, sketch the Cassinian curves |C[z]| = const., then check your answer using a computer. To answer the following questions, recall that if R(z) is a real function of position in the plane, then R(p) is a local minimum of R if R(p) < R(z) for all z in the immediate neighborhood of p. A local maximum is defined similarlty.

(i) Referring to the previous exercise, what is the significance of the orthogonal trajectories of the Cassinian curves you have just drawn?

They have the same significance as in the quadratic Cassini curves (Ex. 6 (iii)). They indicate the lines of steepest ascent and descent.

(ii) Does |C(z)| have any local maxima?

No. |C(z)| is a real runction of position z in the plane. For every point z in $\mathbb{C}$, there is at least one point p in the immediate neighborhood such that |p| > $|z|$, so that $|C(p)| > |C(z)|$. This happens because |C(z)| = $|z - a_1|\,|z - a_2|\,|z - a_3|$. All three terms will amplify with p compared to z. Hence, $|C(z)|$ has no local maximum.

(iii) Does |C(z)| have any non-zero local minima?

The local minima |C(1)|, |C(-1)| and |C(-1-i)| are all zeros. The two saddles are the only points where one might look for a non-zero local minimum. In the saddle points (z_s), there are points in the immediate neighborhood where |C(p)| < |$C(z_s)$|, so |C(z)| at saddle points cannot be local minima. We conclude that |C(z)| does not have a local minimum.

(iv) If D is a disk (or indeed a more arbitrary shape), can the maximum of |C(z)| on D occur at a

point inside D, or must the maximum occur at a boundary point of D? What about the minimum of |C(z)| on D?

The modular surface represents |C(z)| at any point in D, including the boundary. The maximum must always occur at a boundary point of D, for otherwise one could amplify z and achieve a higher point on the modular surface, because |C(z)| = $|z-a_1|\,|z-a_2|\,|z-a_3|$. If z is amplified (|z| increases), all three terms grow, so the product grows. If |C(z)| = const., the maximum may occur inside D as well as on the boundary. By the same reasoning, the minimum of |C(z)| must occur at one or more points inside the boundary unless |C(z)| = const., in which case it coincides with the maximum.

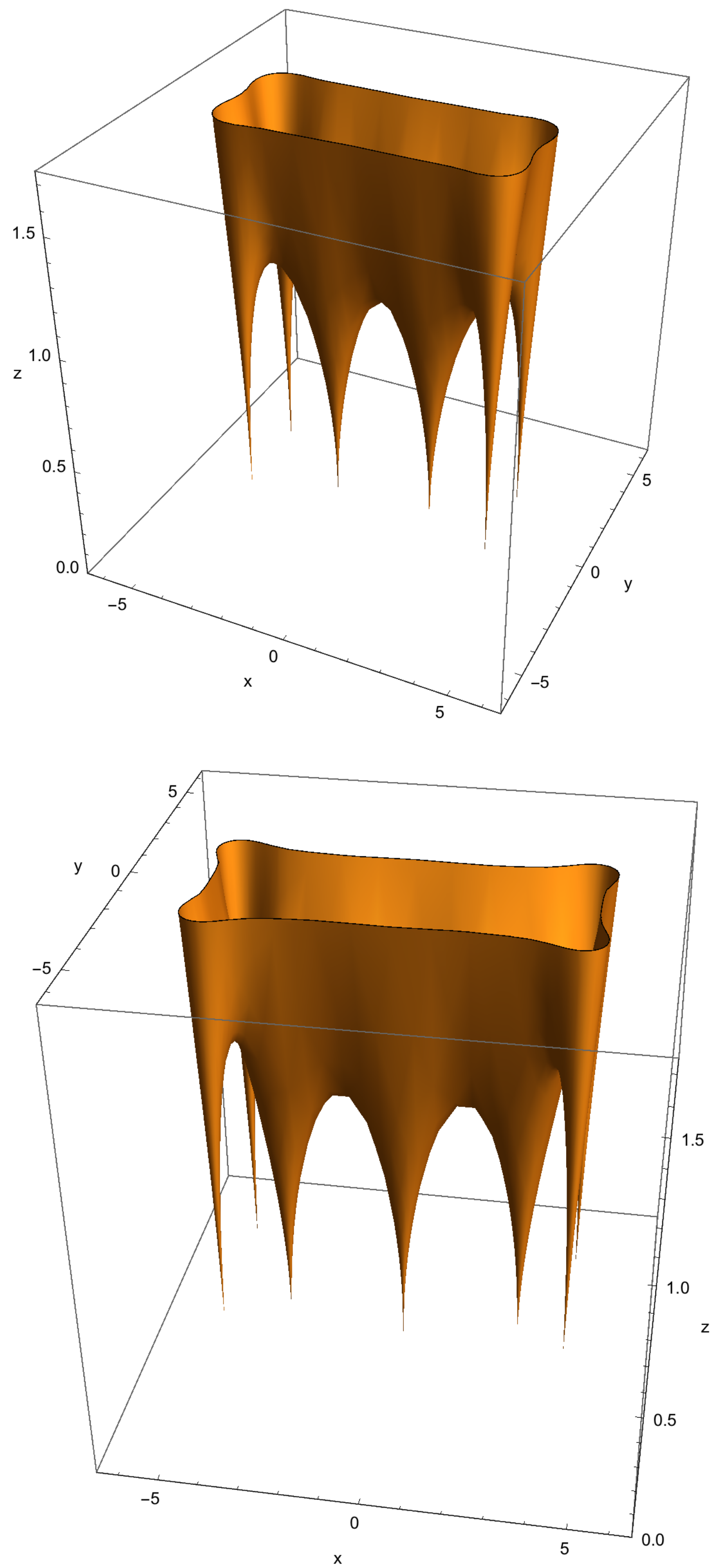

Figure 3 (a, b). Modular surfaces,
Left: $\cos(z) = 1 - z^2/2! + z^4/4! + z^6/6!...$, Right: $\sin(z) = z - z^3/3! + z^5/5! - z^7/7!...$

(v) Do you get the same answers to these questions if C(z) is replaced by an arbitrary polynomial?

By “arbitrary polynomial” Needham was apparently referring to the equation on p. 63.

So long as |z| can increase, there should be no local maxima.

Roots always indicate |C(**a**)| = 0, where **a** is a root, so these are not non-zero local minima.

If D is a bounded region the maxima must occur on the boundaries, or |C(z)| must be a constant and form the entire surface within and including the boundary.

Minima must occur inside D.

What about a complex function that is merely known to be expressible as a power series?

The series for cos z and sin z are used as examples of power series (Figure 3a, b). Cos z = 1 $- \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} +$ .... and sin z = z $- \frac{z^3}{3!} + \frac{z^5}{5} - \frac{z^7}{7!} +$ ... do appear to be interpretable as contours (Figure 3). Since I have not plotted trajectories, there are also no plots of orthogonal trajectories. If there were, they would have the same interpretation as lines of greatest ascent and descent. Figures 3 a and b were plotted with Plot3D using the polynomial format. Multiplying the formulas by *i* did not change the plots.

So long as |z| can increase, there should be no local maxima. If D is a bounded region as shown in Figure 3, the maxima must occur on the boundaries.

Cos(z) shows six local minima corresponding to the 6th power of the truncated series. On side view they appear to be non-zero, but that must be an artifact of the plotting function in *Mathematica*. When z is zero, cos(z) = 1. Since it occurs in the saddle, it cannot be a local minimum or maximum. Sin(z) shows seven local minima corresponding to the 7th power of the truncated series.