Needham, Chapter 2, Exercise 6
G. Palmer, June 20, 2020, 1:45 pm PST

6. This question refers to the Cassinian curves in [9] on page 62.

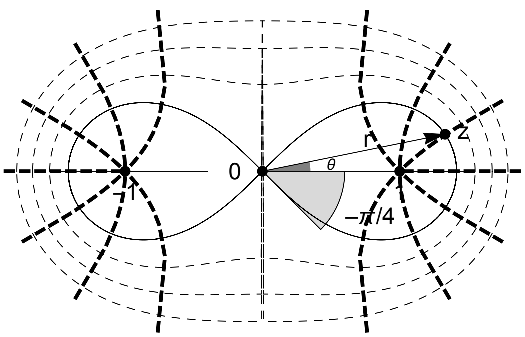

Figure 1a. After [9a], p. 62], z = $\pm\sqrt{w}$

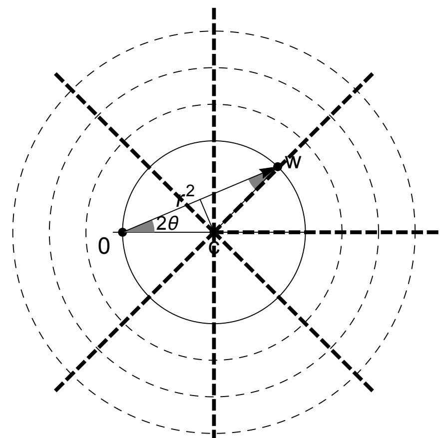

Figure 1b. After [9b], p. 62], w = z^2

(i) On a copy of this figure, sketch the curves that intersect each Cassinian curve at right angles; these are called the orthogonal trajectories of the original family of curves.

See Figure 1a. The curves that intersect each Cassinian curve at right angles are the rays of the circle in the w-plane transformed by w $\longrightarrow \pm\sqrt{w}$. This is justified in part (iv), but included here only for information. I believe that Vasco pointed out in his answer that if you start at a point outside the outer curve in 1a and draw the corresponding curve orthogonal to each succeeding curve, as the lemniscate forms, the orthogonal curve bends towards the focus on the same side of the y-axis. The reason for that is given in (ii).

(ii) Give an argument to show that each orthogonal trajectory hits one of the foci at ± 1.

The rays orthogonal to the circles in figure 1b all intersect at 1, which corresponds to the foci of the lemniscate at 1 and –1 in figure 1a. *The preimages of w = 1 are z = ± 1, so these are the foci* (p. 62, ¶ 1). Hence, we expect all the orthogonal curves of the lemniscates to intersect at one of the two foci.

(iii) If the Cassinian curves are thought of as a geographical contour map of the modular surface (cf. [10]) of $\left(z^2 - 1\right)$, then what is the interpretation of the orthogonal trajectories in terms of the surface?

We may think of the contour lines as representing curves parallel to $\mathbb{C}$ and rising successively in height. On such a curve seen from outside the modular surface at the same level as the curve, the slope of the tangent is zero, and the ascent or descent of any trajectory on the tangent is zero. If we rotate a tangent line at the point of tangency in a plane parallel to the modular surface, it will reach maximal departure from the horizontal when it is orthogonal to the curve. From outside the modular surface at the same level as the curve, or from above, we would see it as a point where an orthogonal trajectory intersects a contour at the angle $\pi/2$. The orthogonal trajectories are therefore the lines of steepest descent or ascent on the modular surface.

(iv) In Chapter 4, we will show that if two curves intersect at some point $p \neq 0$, and if the angle between them at p is ϕ, then the image curves under $z \mapsto w = z^2$ will also intersect at angle ϕ, at the point $w = p^2$. Use this to deduce that as z travels out from one of the foci along an orthogonal trajectory, $w = z^2$ travels along a ray out of $w = 1$.

z travels out from a focus point along a trajectory orthogonal to the lines of the lemniscate as shown in (i) and (ii). In the w plane under z^2 both focal points become $z^2 = 1$ and the contour lines of the lemniscate preimages become concentric circle images with origin at 1, as shown in Exercise 5. Since z is traveling out from one of the foci along an orthogonal trajectory, w must travel out from 1 along an orthogonal trajectory. That is the definition of a ray.

(v) Check the result of the previous part by using a computer to draw the images under $w \mapsto \sqrt{w}$ of (A) circles centered at $w = 1$; (B) the radii of such circles.

See figure 1a.

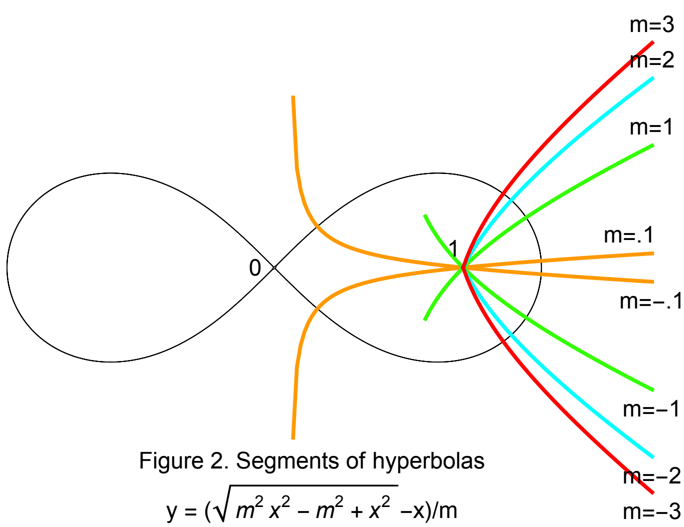

Figure 2. Segments of hyperbolas
$y = (\sqrt{m^2 x^2 - m^2 + x^2} - x)/m$
on Cassinian curve

(vi) Writing $z = x + iy$ and $w = u + iv$, find u and v as functions of x and y. By writing down the equation of a line in the w-plane through $w = 1$, show that the orthogonal trajectories of the Cassinian curves are actually segments of hyperbolas.

Since $w = z^2$, we write

$$(x + \mathrm{i}y)^2 = \mathrm{u} + \mathrm{iv}$$

x^2+2ixy – y^2 = u + iv

Then setting real and imaginary parts equal,

v = 2xy

u = x^2– y^2

[Write] *down the equation of a line in the w-plane through w = 1.*

v = m(u–1)

Substitute for u and v to obtain an equation in the xy-plane.

2 x y = m (x^2– y^2 – 1)

m (x^2– y^2– 1) – 2 x y = 0

This is the equation for a hyperbola in the cartesian system. By the Wikipedia page, $A_{xx}\, x^2 + 2\, A_{xy}\,xy + A_{yy}\, y^2 \; + \; 2\, B_x\, x \; + \; 2\, B_y\, y + $C = 0 is a hyperbola provided that

$$D := \begin{vmatrix} A_{xx} & A_{xy} \\ A_{xy} & A_{yy} \end{vmatrix} < 0$$

(https://en.wikipedia.org/wiki/Hyperbola#Conjugate_hyperbola)

In this case, we have

$$D := \begin{vmatrix} m & -2 \\ -2 & -m \end{vmatrix} = -\, m^2 - 4 < 0$$

The equation has the solutions

$$\left\{\left\{y \to \frac{-\sqrt{m^2\, x^2 - m^2 + x^2}\; - x}{m}\right\}, \left\{y \to \frac{\sqrt{m^2\, x^2 - m^2 + x^2}\; - x}{m}\right\}\right\}$$

which give y = 0 when x = 0 and y = {–2/m, 0} when x = 1. Segments based on the second solution with several values of m are shown in Figure 2.

Another try:

(w - 1) is a vector with tail at 1. Let a be a arbitrary real number that expands or contracts the length of the line l(w).

$$l(w) = a(w - 1) + 1 = a(u + iv - 1) + 1$$

Substitute for u and v.

$$= a(x^2 + 2ixy - y^2 - 1) + 1$$

$$= a[(x^2 - y^2 - 1) + i2xy] + 1$$

$$= a(x^2 - y^2 - 1) + 1 + i2axy$$

This is another quadratic equation much like the one above. It does plot hyperbolas in ContourPlot.

$$D := \begin{vmatrix} a & 2 \\ 2 & -a \end{vmatrix} = -a^2 - 4 < 0$$

The determinant shows that the equation describes a hyperbola in the cartesian system.

Since by (v), curves that intersect in the w-plane have identical angles of intersection in the z-plane, and since rays through 1 in the w-plane are orthogonal to circles centered at 1 in the w-plane, and since our two equations for a line passing through 1 in the w-plane both produced equations for hyperbolas in the xy-plane, we conclude that the orthogonal trajectories of the Cassinian curves are actually segments of hyperbolas.