Needham, Chapter 2, Exercise 5, P. 111.
G. Palmer, June 14, 2020, 2 pm PST

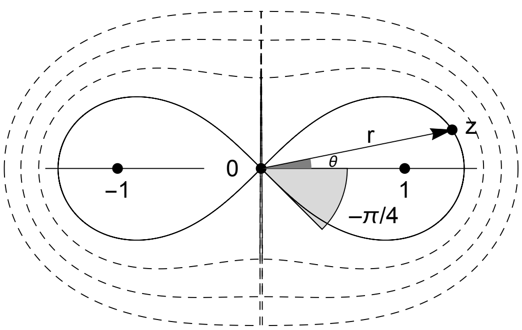

Figure 1a. After [9a], p. 62], $\sqrt{2\cos 2\,\theta}\; e^{i\theta}$

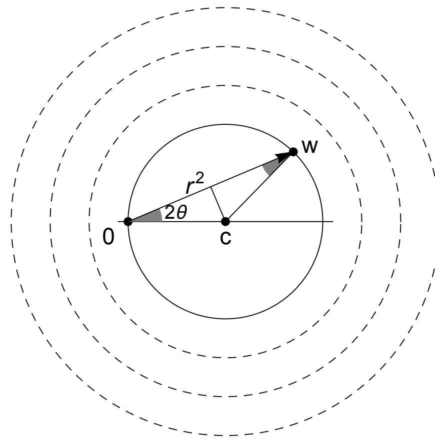

Figure 1b. After [9b], p. 62], w = z^2

5 Show that the mapping z -> z^2 doubles the angle between two rays coming out of the origin. Use this to deduce that the lemniscate (see [9] on p. 62) must self-intersect at right angles.

Let the angles of the two rays be θ and ϕ.

If $z_1 = \mathrm{r}e^{i\theta}$, then $z_1^2 = \mathrm{r}e^{i\theta} \cdot \mathrm{r}e^{i\theta} = r^2\, e^{i2\theta}$
If $z_2 = \mathrm{R}e^{i\phi}$, then $z_2^2 = R^2\, e^{2\,i\phi}$

$$\mathrm{Arg}(z_1) - \mathrm{Arg}(z_2) = \theta - \phi$$

$$\mathrm{Arg}\left(z_1^2\right) - \mathrm{Arg}\left(z_2^2\right) = 2\theta - 2\phi = 2(\theta\text{-}\phi) = 2\;(\mathrm{Arg}(z_1) - \mathrm{Arg}(z_2))$$

This shows that the mapping z $\longrightarrow z^2$ doubles the angle between two rays coming out of the origin.

The angle θ shown in [9a] is half the angle under z^2 shown in [9b]. As w traverses the circle of [9b] in an anti-clockwise direction and approaches the origin, its angle 2θ with the x axis approaches $\pi/2$ in the limit. Hence, the angle θ in the lemniscate [9a] must approach $\pi/4$ as z approaches the origin. As w traverses in the clockwise direction and approaches the origin, its angle with the x axis approaches $-\pi/2$ in the limit. Hence, the θ in [9a] must approach $-\pi/4$. Since the the lobe on the LHS of the lemniscate in [9a] has “limbs” (nearly straight segments of the curve) intersecting at the origin, and those limbs are symmetrical to those on the RHS by f(θ) = $\sqrt{2\cos(2\,\theta)}\;\, e^{i\theta}$, their angles with the x axis must also be multiples of $\pi/4$. In fact, they are $3\pi/4$ and $5\pi/4$ (= $-3\pi/4$).* (Another way to see this would be to place c at (–1) in [9b] and repeat the approaches to 0.) We have established that the four angles of the lemniscate occurring on either side of the x-axis equal $\pi/4$, the same must be true of the four angles on either side of the y-axis. The angle between any two adjacent limbs of the lemniscate that intersect at the origin must be $\pi/2$. Therefore, the lemniscate must self-intersect at right angles.

*Or one can write z = $\pm\sqrt{1+e^{i\phi}}$, where ϕ = arg(z-c). One can also write $(x-1)^2 + y^2 = 1$, y = $\pm\sqrt{2x-x^2}$, w = x $\pm i\sqrt{2x-x^2}$, and z = $\pm\sqrt{x \pm i\sqrt{2x-x^2}}$, where x in [0..2]. All the equations are variations of z = $\pm\sqrt{w}$.