Needham, Ch. 2, Ex. 43, p. 121.

G. Palmer, March 5, 2021, 1:40 pm PDST

43. Use Gauss' Mean Value Theorem [p. 110] to find the average of cos z over the circle |z| = r. Deduce (and check with a computer) that for all real values of r,

$$\int_0^{2\pi} \cos[r\cos\theta] \times \cosh[r\sin\theta]\, d\theta = 2\pi.$$

Gauss's Mean Value Theorem:

$$\langle f(z) \rangle_C = \frac{1}{2\pi}\int_0^{2\pi} f\left(k + \mathrm{R}e^{i\theta}\right) d\theta = \mathrm{f(k)}$$

$$\mathrm{f(z) = \cos z}$$

|z| = r is centered on the origin, so

$$\langle f(z) \rangle_C = \frac{1}{2\pi}\int_0^{2\pi} f\left(k + r\,e^{i\theta}\right) d\theta = \mathrm{f(k)} = \mathrm{f(0)} = \cos 0 = 1$$

and k = 0, so

$$\frac{1}{2\pi}\int_0^{2\pi} f\left(\mathrm{r}e^{i\theta}\right) d\theta = 1$$

$$\frac{1}{2\pi}\int_0^{2\pi} f\left(\mathrm{r}e^{i\theta}\right) d\theta = \frac{1}{2\pi}\int_0^{2\pi} \cos\left(\mathrm{r}e^{i\theta}\right) d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos(r\,(\cos\theta + i\sin\theta))\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos(r\cos\theta + i\,r\sin\theta)\, d\theta$$

Now we use cos(x + iy) = cos x cosh y - i sin x sinh y from p. 89.

$$= \frac{1}{2\pi}\int_0^{2\pi} [\cos(r\cos\theta)\cosh(r\sin\theta) - i\sin(r\cos\theta)\sinh(r\sin\theta)]\, d\theta = 1$$

Equate real and imaginary parts:

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos(r\cos\theta)\cosh(r\sin\theta)\, d\theta = 1$$

$$\Longrightarrow \int_0^{2\pi} \cos(r\cos\theta)\cosh(r\sin\theta)\, d\theta = 2\pi$$

This is what we wanted.

Using the Integrate[] function, *Mathematica* required 1.5 to 2 minutes to compute 37 integrals with varying values of r, but the results support the conclusion that the equation works for all real values of r. One could compute all the values in less than one second with NIntegrate[], but it fails to converge beyond r = 8π.

```
In[193]:= Clear["Global`*"];

time = AbsoluteTiming@
   Table[Integrate[TrigToExp[Cos[r Cos[θ]] Cosh[r Sin[ θ]]], {θ, 0, 2 Pi}],
    {r, 0, 12 Pi, Pi/3}];

Print["results over ", Length @ time[[2]],
  " real values r = (0, π/3, 2π/3, ..., 12π)", time[[2]],
  "\ncalculation time: ", time[[1]], " seconds"];
```

```
results over 37 real values r = (0, π/3, 2π/3, ..., 12π)
{2 π, 2 π, 2 π, 2 π, 2 π, 2 π, 2 π, 2 π, 2 π, 2 π, 2 π, 2 π, 2 π, 2 π, 2 π, 2 π, 2 π, 2 π, 2 π,
 2 π, 2 π, 2 π, 2 π, 2 π, 2 π, 2 π, 2 π, 2 π, 2 π, 2 π, 2 π, 2 π, 2 π, 2 π, 2 π, 2 π, 2 π}
calculation time: 89.6557 seconds
```