Needham, Ch. 2, Ex. 42, p. 121.

G. Palmer, March 4, 2021, 3:20 pm PDST

42. Consider the mapping $z \mapsto w = P_n(z)$, where $P_n(z)$ is a general polynomial of degree $n \geq 2$. Let S_q be the set of points in the z-plane that are mapped to a particular point q in the w-plane. Show that the centroid of S_q is independent of the choice of q, and is therefore a property of the polynomial itself. [Hint: This is another way of looking at a familiar fact about the sum of the roots of a polynomial.]

The solution to this problem escaped me the first and second times I tried it. At first I could not see what the sum of the roots had to do with the centroid. Then, when Vasco pointed out in personal communication that one could write a second polynomial $(P - q)$, I could not see what this polynomial, which he called "Q", had to do with the original P. But it eventually dawned on me that since $q = P_n(z_j)$, where z_j in S_b, and $Q_n(z_j) = (P_n(z_j) - q) = 0$, then the points z_j of $Q_n(z_j)$ are the same as those mapped by P to q. The roots of $Q_n(z)$ satisfy both equations. Vasco pointed out that this is true by definition. For me, it was a blind spot. Then one must find a centroid that makes use of the sum of roots and inspect it to determine whether it depends on q.

We can write an equation for a general polynomial in terms of powers and coefficients of z.

$$P_n(z) = a_n z^n + a_{n-1} z^{n-1} + \cdot\cdot\cdot + a_1 z + a_0 = \mathrm{q} \qquad (1)$$

for any z in $S_q = \{z_j\}$, where q is an arbitrary complex number. Now if we write $(P_n(z_j) - q) = 0$, we see that the z_j's must be those of S_q, the same z_j's for which $P_n(z_j) = q$. Vasco writes $Q_n(z_j) = (P_n(z_j) - q)$. Then since Q_n equals 0, it can be factored. The roots constitute S_q and satisfy both P and Q. Write the factored form of $Q_n(z)$, with roots which are not necessarily unique.

$$Q_n(z) = (z - C_n)(z - C_{n-1}) \cdot\cdot\cdot (z - C_1) = 0 \qquad (2)$$

By Vieta's formula for the sum of the roots of a polynomial (https://en.wikipedia.org/wiki/Vieta%27s_formulas), we use the coefficients from (1) and the roots from (2) to write:

$$\sum_{j=1}^{n} C_j = -(a_{n-1})/a_n \qquad (3)$$

Since

$$Q_n(z) = P_n(z) - q = a_n z^n + a_{n-1} z^{n-1} + \cdot\cdot\cdot + a_1 z + a_0 - q$$

the centre of mass formula on p. 103 applied to the centroid of the set of particles $\{z_j\}$ of equal mass may be used with (3) to write the centroid of S_q:

$$Z = \frac{1}{n} \sum_{j=1}^{n} C_j = -\frac{1}{n}(a_{n-1})/a_n$$

Depending only on the number of roots and the n^{th} and $(n-1)^{\text{th}}$ coefficients of $P_n(z)$, Z is independent of q.