Needham, Ch. 2, Ex. 41, p. 121.
G. Palmer, March 28, 2021, 5:40 pm PDT

Out[∘]=

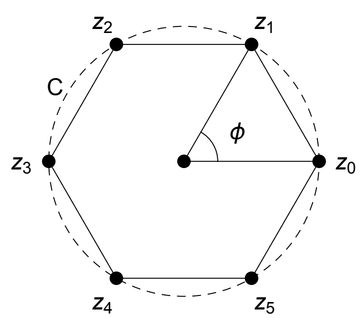

Figure 1a. Regular hexagon

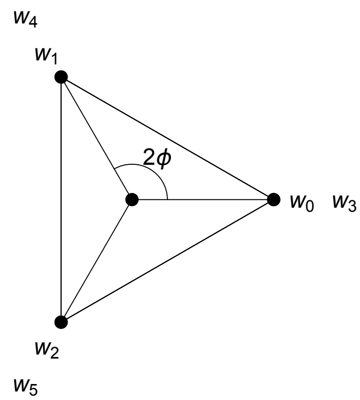

Figure 1b. Hexagon under $w = z^2$

41. To establish (36) let z_0, z_1, z_2, ..., z_{n-1} be the vertices (labelled counterclockwise) of a regular n-gon, and let C be the circumscribing circle. Also, let $w_j = z_j^m$ be the image of vertex z_j under the mapping $z \mapsto w = z^m$ [changed from $z \mapsto z = z^m$––– GBP]. *Think of z as a particle that starts at z_0 and orbits counterclockwise round C, so that the image particle $w = z^m$ starts at w_0 and orbits round another circle with m times the angular speed of z.*

(36) *Unless m is a multiple of n, the image under z^m of an origin-centred regular n-gon is an origin-centred regular N-gon, where N = (n divided by the highest common factor of m and n). If m is a multiple of n, then the image is a single point.* (p. 106)

(i) Show that each time z travels from one vertex to the next, w executes (m/n) of a revolution. Thus as z travels from z_0 to z_k, w executes k(m/n) revolutions as it travels from w_0 to w_k.

Let C be the circumscribing circle of radius r, and let $z_0 = r$. Let $z_1 = re^{i\phi}$, where $\phi = 2\pi/n$ is the argument of z_1. Since this is a regular polygon, ϕ is the radial distance traveled each time z travels from one vertex to the next. Then because $w = z^m$, arg(w) = m ϕ = (m/n) 2π, which is (m/n) of a revolution, is the radial distance traveled each time w travels from one vertex to the next. It follows that as *z travels from z_0 to z_k*, it moves k/n revolutions, while w executes k(m/n) revolutions as it travels from w_0 to w_k.

(ii) Let w_k be the first point in the sequence w_1, w_2, etc., such that $w_k = w_0$. Deduce that if (M/N) is (m/n) reduced to lowest terms, then k = N. Note that N = (n divided by the highest common factor of m and n).

We assume the construction is as illustrated in Figure 1, except that we find the lowest terms of m/n where m = 2 and n = 6 to be M = 1 and N = 3. Hence, $w_3 = w_0$ and we can ignore higher numbers of k.

Substitute M and N for m and n. We know from (i) that w executes k(M/N) revolutions as it travels from w_0 to w_k. Since it is given that $w_k = w_0$, then w has returned to its starting point in k steps. Since this is a regular n-gon, w must return to its starting position in M revolutions.

$$M = k(M/N) \Longrightarrow k = N$$

(iii) Explain why $w_{N+1} = w_1$, $w_{N+2} = w_2$*, etc.*

In part (ii) we showed that k = N where $w_k = w_0$. Then,

$$w_{N+1} = w_{k+1} = w_{0+1} = w_1$$

$$w_{N+2} = w_{k+2} = w_{0+2} = w_2$$

…

(iv) Show that $w_0, w_1, \ldots, w_{N-1}$ *are distinct.*

We need only show that none of the values can repeat in N–1 steps.

In part (ii) and (iii) we showed that w returns to its starting position only after M revolutions when k = N and each step is k(M/N). That means that during the M revolutions, it could not have landed on its starting position. Since all steps are equal, none of the values of steps k = 1 to N repeat within the M revolutions of w_N. Therefore, $w_0, w_1, \ldots, w_{N-1}$ are distinct.

Out[ ]=

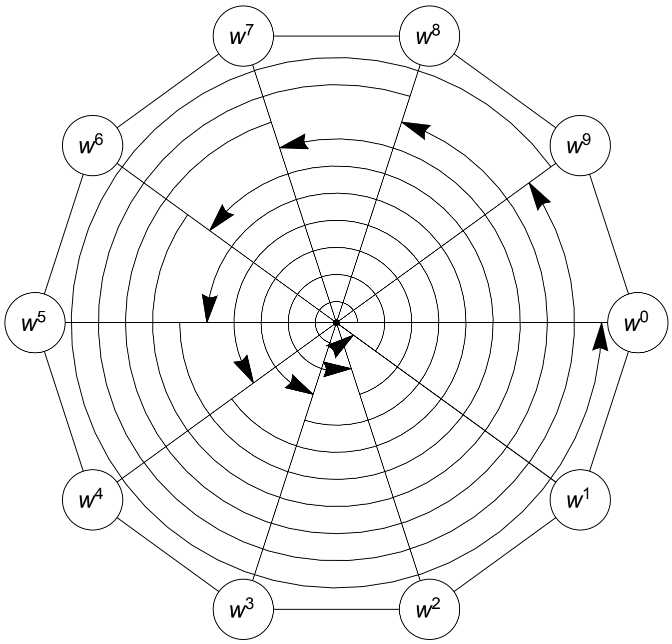

Figure 2. 10-gon under w = z^9
Each angle between rays is $2\pi(M/N)$
Each step moves an arrow 9/10 2π

(v) *Show that w_0, w_1, ..., w_{N-1} ar*e the vertices of a regular N-gon.

In (iv) it was established that the N values w_0, w_1, ..., w_{N-1} are distinct, in (ii) that as z makes one revolution in n steps ending at z_n, z^m makes a whole number M of revolutions in N steps (M and N as defined above) ending at w_0, and that the w_j are uniformly spaced at (M/N) revolutions. This means that in the space of M revolutions, one can plot N points w_j spaced at equal angles around 0.* For example, suppose that M/N = 9/10, as in Figure 2. Each arrow ends at the terminating angle of the traversal of $w = z^9$ under one revolution of z on a 10-gon. Each arrow in succession (from center outwards) starts at the angle where the previous traversal of w finished. Even though the revolutions of w are anticlockwise, the order in which the w_i are plotted is clockwise, due to the fact that each revolution of z produces a 9/10 revolution of z^9. The total number of steps taken by w equals 10 in this case, but the total number of revolutions around the origin is 9, the number of the power in the reduced fraction.

Where j = 0..(N-1), we can write the set of points of w as $w_j = w_0\, e^{i2\pi(M/N)j}$. From the uniqueness and equal spacing (in spite of the reversal of order), one can deduce that the w_j are the roots of unity $e^{i2\pi(M/N)j}$, j = 0..(N-1) multiplied by w_0.** Hence, the w_j form the basis for an origin-centred N-gon. If m is a multiple of n, then M/N is a whole number and $w_j = w_0\, e^{i2\pi(M/N)j}$ multiplies 2π by a whole number in the exponent of e, so the image is a single point. This establishes (36).

*Note: As shown by the example, this does not mean that the w_0, w_1, ..., w_{n-1} necessarily appear in anticlockwise order of progression around the N-gon as the example shows.

** For the purpose of this exercise, we pay no attention to the conjugate pairs.