Needham, Chapter 2, Exercise 4, P. 111.
G. Palmer, June 4, 2020, 11:40 am PST

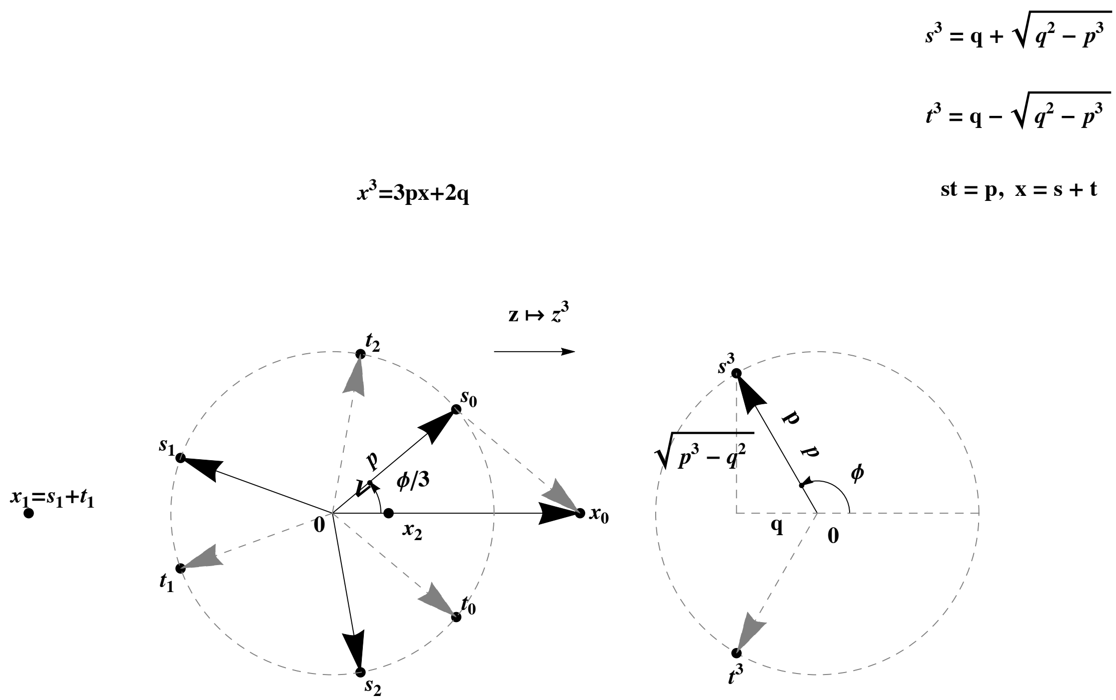

Figure 1. Solution to general cubic equation
using z $\mapsto$ z^3 mapping and Cardano's formula.
$q^2 \leq p^3$

4. In figure [7] [p. 60] we saw that if $q^2 \leq p^3$ then the solutions of $x^3 = 3px + 2q$ are all real. Draw the corresponding picture in the case $q^2 > p^3$, and deduce that one solution is real, while the other two form a complex conjugate pair.

Givens: x^3 + 3px + 2q, $q^2 > p^3$, st = p, x = s + t

Figure 1 reproduces the essentials of Figure [7], p. 60 pertaining to $q^2 \leq p^3$. But if $q^2 > p^3$, Viete's formula will not work, but Cardano's still applies, so $s^3 = q + \sqrt{q^2 - p^3}$ and $t^3 = q - \sqrt{q^2 - p^3}$, where p and q are real. Both q^2 and p^3 may be positive or negative. Vasco has provided a clear answer to this problem. I will try to write a different solution. Due to the complications within the RHS of s^3 and t^3, I had a difficult time getting perspective on this problem, but it turned out to be relatively simple.

Ignore the p's and q's for the moment and begin with the information on using the regular n-gon to find roots as explained in Section 3, pp. 57-78. We use it to find three roots of s^3. Then one can apply t = p/s and z = s + t to obtain the three solutions, which depend in a somewhat complicated way on the absolute magnitudes of p and q and their signs. To apply the regular n-gon approach, one needs the angle ϕ, which is here defined as arg(s^3). Given that p and q are both real, we find that ϕ equals 0 or π depending on the sign of s^3. Following the general approach in Section 3, we write s $= |s| e^{i(\phi/3+n2\pi/3)}$, where $\phi = \arg(s^3)$, $s^3 = q + \sqrt{q^2 - p^3}$, and n = {0, 1, 2}. In Figures 2 and 3, the dashed circles are the unit circle and the dotted circles have radius |s|.

Case 1, $s^3 > 0$.

$s^3 > 0$ when q > 0 or when q < 0 and p < 0.

When $s^3 > 0$, $\phi = 0$, so $s_n = \{|s|, |s|e^{i(2\pi/3)}, |s| e^{i(4\pi/3)}\}$, where $|s| = \sqrt[3]{s^3}$. Of the s_n, s_0 is real; s_1 and s_2 form a conjugate pair. Knowing that st = p, we can calculate three values for $t_n = p / s_n$. Then we can calculate three solutions $z_n = s_n + t_n$. The first, z_0, must be real because its calculation ($z_0 = s_0 + t_0$ where $t_0 = p/s_0$) involves only real numbers. Note also that since $s_2 = \overline{s_1}$ and p is real, then $t_2 = \overline{t_1}$ and $z_2 = \overline{z_1}$. Since p may be either positive or negative, one could define two subcases, but there seems no need to do so. If $t_2 = \overline{t_1}$, then $-t_2 = -\overline{t_1}$. This reasoning does not apply to the second case where p > 0. Hence, whether p > 0 or p < 0, it is still the case that there is one real solution and a conjugate pair of solutions.

$$s_0 = |s| e^{i0} = |s|, \quad t_0 = p/s_0, \quad z_0 = |s| + t_0 \text{ (real)}$$
$$s_1 = |s| e^{i2\pi/3}, \quad t_1 = p / s_1 = p / s_1, \quad z_1 = s_1 + t_1$$
$$s_2 = |s| e^{i4\pi/3} = \overline{s_1}, \quad t_2 = p / s_2 = p / \overline{s_1} = \overline{t_1}, \quad z_2 = \overline{s_1} + \overline{t_1} = \overline{z_1}$$

Case 2, $s^3 < 0$.

$s^3 < 0$ when q < 0 and p > 0.

When $s^3 < 0$, $\phi = \pi$, so $s_n = \{-|s|, |s| e^{i(5\pi/3)}, |s|e^{i(\pi/3)}\}$.

$$s_0 = |s| e^{i\pi} = -|s|, \quad t_0 = p/s_0, \quad z_0 = s_0 + t_0 \text{ (real)}$$
$$s_1 = |s| e^{i5\pi/3}, \quad t_1 = p / s_1, \quad z_1 = s_1 + t_1$$
$$s_2 = |s| e^{i\pi/3} = \overline{s_1}, \quad t_2 = p / s_2 = \overline{t_1}, \quad z_2 = \overline{s_1} + \overline{t_1} = \overline{z_1}$$

In both cases, there is one real solution and a conjugate pair. Hence, we deduce that where $q^2 > p^3$, one solution is real, while the other two form a complex conjugate pair.

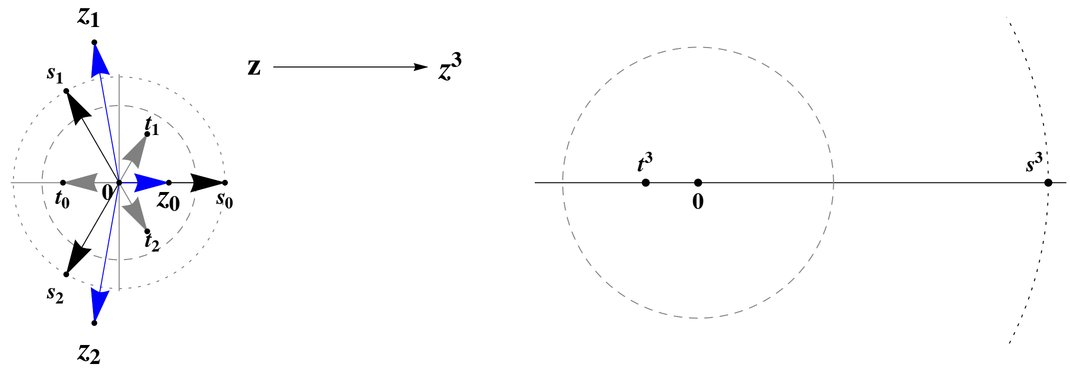

Figure 2. Solutions to reduced general cubic, Cardano's formula.
$q^2 > p^3$, $s^3 > 0$, p = −1, q = 1.1

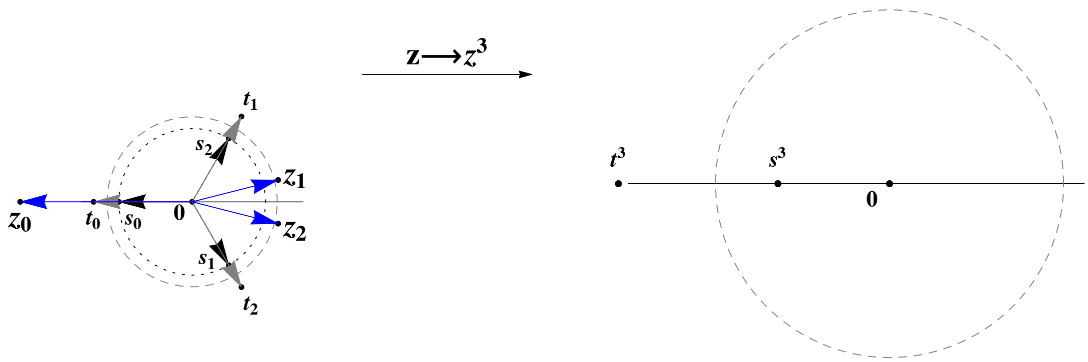

Figure 3. Solutions to reduced general cubic Cardano's formula.
$q^2 > p^3$, $s^3 < 0$, p = 1, q = −1.1