Needham, Chapter 2, Exercise 39, p. 121

G. Palmer, February 9, 2021 4 pm PDST

Out[1565]=

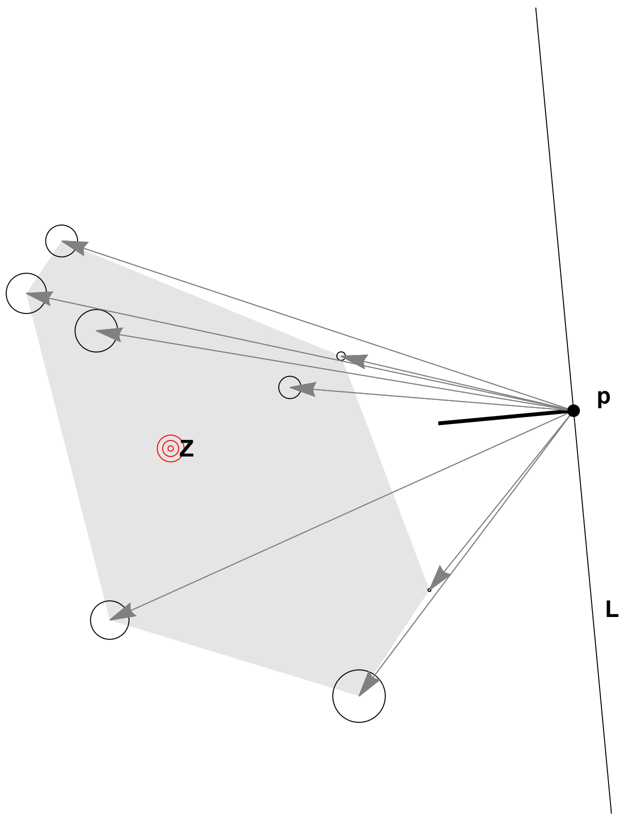

Figure 1. Random weighted points
Centroid must be inside convex hull.

39. Show that (32) is still true even if the (positive) masses of the particles are not all equal.

(32) The centroid Z must lie in the interior of the convex hull H.

The centroid of a set of n point-particles in $\mathbb{C}$ is defined on p. 103 as

$$Z = \frac{\sum_{j=1}^{n} m_j z_j}{\sum_{j=1}^{n} m_j}$$

It is a condition of the centroid with particles of equal mass that $\Sigma(z_j - Z) = 0$, where $(z_j - Z)$ describes one of the n vectors from Z to the point z_j in or on the convex hull. We take $m_j \ (z_j - Z)$ to describe a vector in a set of particles having mass. Then it should be the case that $\Sigma m_j (z_j - Z) = 0$. For example, what if five very light particles are distributed on the west side of the y-axis and one heavy particle is located on the east side? It is possible that they could sum to zero given the right complex position vectors. Without knowing any specific masses or locations, is it still possible to show that the centroid must lie within the complex hull?

This is a proof by contradiction. Suppose the centroid is exterior to the convex hull, say at a point p. Draw a line L passing through p so that all points on or inside the hull lie on one side of L. The figure can now be rotated and translated without loss of generality. Rotate the plane about p, so that the orthogonal to L at p (thick black line in Figure 1) is horizontal, and translate the plane so that p = 0. Then it is clear that all the vectors from p to points on or in the hull have a real component in the same direction on the real axis. These components, all having positive mass, would all have the same sign and therefore the projected vectors would not sum to zero, and p cannot be the centroid.

If the centroid candidate p were a point on the convex hull we can again draw a line L through p so that all other points on or inside the hull fall on one side of L and have components on the line orthogonal to L. This p on the hull would be a zero vector, no matter what it's mass. Reasoning as above, the masses of the remaining particles are all positive by definition, so it is not possible for the points to sum to zero.

Hence, the centroid Z must be interior to the convex hull.

Figure 1 scatters 8 random points of randomly varying masses indicated by size of radius and draws a line L outside the convex hull at an arbitrary angle that does not permit it to intersect the hull. The red circles in the hull are centred on the actual calculated centroid. The thick black line is orthogonal to L.