Needham, Chapter 2, Exercise 38, p. 120-121

G. Palmer, February 6, 2021, 8:30 am PDST

38. As in the previous exercise, let Θ= Arg(z).

(i) Use (29) to show that

$$\frac{1}{2} Log\left[\frac{1+z}{1-z}\right] = z + \frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7} + \ldots$$

(29) $\text{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \ldots$

$$\text{Log}(1 - z) = - z - \frac{(-z)^2}{2} + \frac{(-z)^3}{3} - \frac{(-z)^4}{4} + \frac{(-z)^5}{5} - \ldots$$

$$= - z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \frac{z^5}{5} - \ldots$$

$$\frac{1}{2} \text{Log}\left[\frac{1+z}{1-z}\right] = \frac{1}{2}(\text{Log}[1+z] - \text{Log}[1-z])$$

$$= \frac{1}{2} ((z - (-z) \; - \frac{z^2}{2} - (- \frac{z^2}{2}) + \frac{z^3}{3} - (- \frac{z^3}{3}) - \frac{z^4}{4} - (- \frac{z^4}{4}) + \frac{z^5}{5} - (- \frac{z^5}{5}) - \ldots$$

$$= \frac{1}{2} (2 z + 0 + \frac{2 z^3}{3} + 0 + \frac{2 z^5}{5} + 0 + \frac{2 z^7}{7} + \ldots)$$

$= z + \frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7} + \ldots$ This is what we wanted.

Out[ ]=

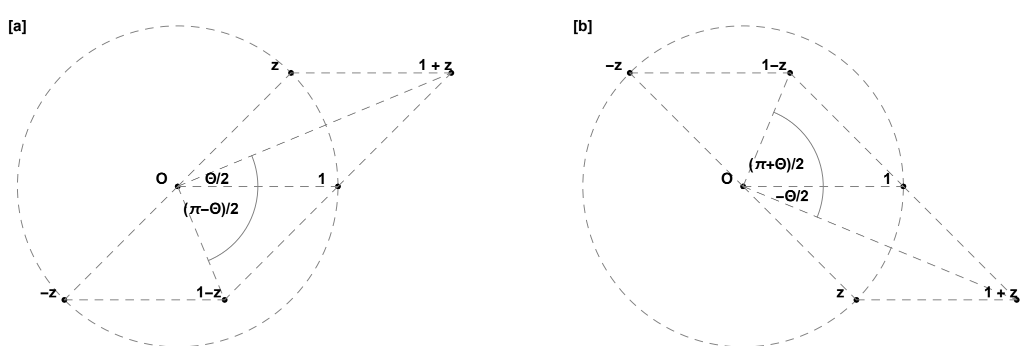

Figure 1. $\text{Im}\{\frac{1}{2} \text{Log}\left[\frac{1+z}{1-z}\right]\}$ = (sign of Θ) [$\frac{\text{Pi}}{4}$]

(ii) Show geometrically that as $z = e^{i\theta}$ goes round and round the unit circle,

$Im\{\frac{1}{2} Log\left[\frac{1+z}{1-z}\right]\}$ = (sign of Θ) $\left[\frac{\pi}{4}\right]$.

In Ex. 37, we determined that Arg[1+z] = Θ/2, where Θ/2 is the argument of the principal value of z + 1. With the denonimator (1-z) we can form a new parallelogram (Figure 1) that is congruent to the parallelogram formed on the numerator (z + 1). It is easily determined that |Arg[1-z]| = (π - |Θ|)/2. This holds for all values of z. Subtract iArg[1-z] from iArg[1+z], and take the imaginary value.

$$\mathrm{Im}\left\{\frac{1}{2}\mathrm{Log}\left[\frac{1+z}{1-z}\right]\right\} = \mathrm{Im}\left\{\frac{1}{2}\left(\ln|1+z| + i\,\mathrm{Arg}(1+z) - \ln|1-z| - i\mathrm{Arg}(1-z)\right)\right\} = \frac{1}{2}(\mathrm{Arg}(1+z) - \mathrm{Arg}(1-z))$$

With a positive sign on Θ, from Figure 1[a],

$$\mathrm{Im}\left\{\frac{1}{2}\mathrm{Log}\left[\frac{1+z}{1-z}\right]\right\} = \frac{1}{2}\left(\frac{\Theta}{2} - \left(-\frac{\pi-\Theta}{2}\right) = \frac{1}{2}\left(\frac{\Theta}{2} + \frac{\pi}{2} - \frac{\Theta}{2}\right) = \frac{\pi}{4}\right.$$

With a negative sign on Θ, from Figure 1[b],

$$\mathrm{Im}\left\{\frac{1}{2}\mathrm{Log}\left[\frac{1+z}{1-z}\right]\right\} = \frac{1}{2}\left(\frac{-\Theta}{2} - \frac{\pi-\Theta}{2}\right) = \frac{1}{2}\left(\frac{-\Theta}{2} - \frac{\pi}{2} + \frac{\Theta}{2}\right) = -\frac{\pi}{4}$$

(iii) Consider the periodic "square wave" function G(θ) whose graph is shown below. Use the previous two parts to deduce that its Fourier series is

$$G(\theta) = \sin\theta + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \frac{\sin 7\theta}{7} \ldots.$$

The graph is a plot of Im{$\frac{1}{2}$ Log$\left[\frac{1+z}{1-z}\right]$} = (sign of Θ) $\left[\frac{\pi}{4}\right]$, which also cycles between π/4 and $-\pi$/4 with every revolution of z around the unit circle. Substitute $e^{i\theta}$ in the logarithmic series and use Euler's equation as above:

$$\frac{1}{2}\mathrm{Log}\left[\frac{1+z}{1-z}\right] = z + \frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7} + \ldots$$

$$= e^{iz} + \frac{e^{3i\theta}}{3} + \frac{e^{5i\theta}}{5} + \frac{e^{7i\theta}}{7} + \ldots$$

$$\mathrm{Im}\left[\frac{1}{2}\mathrm{Log}\left[\frac{1+z}{1-z}\right]\right] = \mathrm{Im}\left[(\cos\theta + i\sin\theta) + \frac{\cos 3\theta + i\sin 3\theta}{3} + \frac{\cos 5\theta + i\sin 5\theta}{5} + \frac{\cos 7\theta + i\sin 7\theta}{7} + \ldots\right]$$

$$= \sin\theta + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \frac{\sin 7\theta}{7} + \ldots$$

Finally, repeat parts (iii) and (iv) of the previous exercise.

(iii) Check this Fourier series by directly evaluating the integrals (19).

$$b_n = \frac{1}{\pi}\int_0^{2\pi} F(\theta)\sin n\theta\, d\theta$$

$$= \frac{2}{\pi}\int_0^{\pi} (\text{sign}\,\Theta)\left(\frac{\pi}{4}\right)\sin n\,\Theta\, d\Theta$$

$$= \frac{-\text{sign}\,\Theta}{2n}\cos n\Theta \,\Big|_0^{\pi}$$

$$b_1 = \frac{-\text{sign}\,\Theta}{2}\cos\Theta \,\Big|_0^{\pi} = \frac{-\text{sign}\,\Theta}{2}(-1 -1) = (\text{sign}\,\Theta)\,1$$

$$b_2 = \frac{-\text{sign}\,\Theta}{2\cdot 2}(\cos 2\pi - \cos 0) = \frac{-\text{sign}\,\Theta}{4}(1 - 1) = 0$$

$$b_3 = \frac{-\text{sign}\,\Theta}{2\cdot 3}(\cos 3\,\pi - \cos 0) = \frac{-\text{sign}\,\Theta}{6}(-1 - 1) = (\text{sign}\,\Theta)\,1/3$$

$$b_4 = \frac{-\text{sign}\,\Theta}{2\cdot 4}(\cos 4\,\pi - \cos 0) = \frac{-\text{sign}\,\Theta}{8}(1 - 1) = 0$$

$$b_5 = \frac{-(\text{sign}\,\Theta)}{2\cdot 5}(\cos 5\,\pi - \cos 0) = \frac{-(\text{sign}\,\Theta)}{10}(-1 - 1) = (\text{sign}\,\Theta)\,1/5$$

The coefficients confirm the series for both positive and negative values of Θ, because $\sin(-\theta) = -\sin\theta$.

Out[ ]=

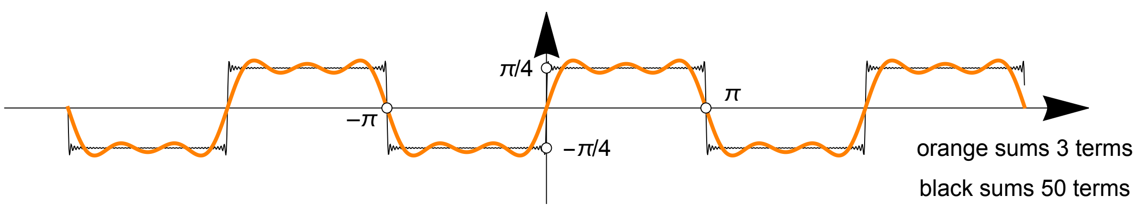

Figure 2. Smooth waves converge to square graph

(iv) Use a computer to draw graphs of the partial sums of the Fourier series. As you increase the number of terms, observe the magical convergence of this sum of smooth waves to the [square] graph above (Figure 2).