Needham, Chapter 2, Exercise 37, p. 120
G. Palmer, February 3, 2021, 5:15 pm PDST

Out[1323]=

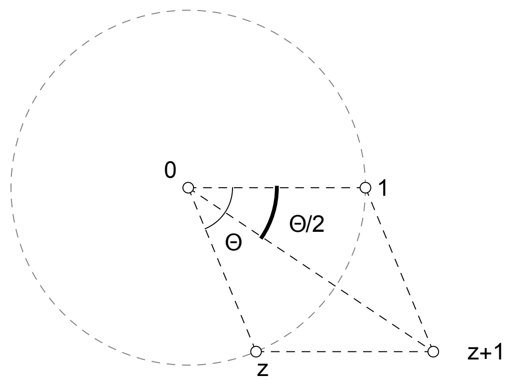

Figure 1. Principal angles in rhombus on unit circle

37. (i) Show geometrically that as $z = e^{i\theta}$ goes round and round the unit circle (with ever increasing θ), Im[Log(1+z)] = (Θ/2), where Θ is the principal value of θ, i.e., -Pi < $\Theta \leq$ Pi.

As z goes round, it makes a rhombus 0-z-(z+1)-1-0 (Figure 1). Here the hyphens represent connecting lines. Taking only the principal values of angles, the equal angles created by the line from 0 to z + 1, bisecting the angles 1-0-z and z-(z+1)-1, show that argument of (z+1) is always half the argument of z.

$$\text{Arg}[z + 1] = \text{Arg}\left[e^{i\Theta}\right]/2 = \Theta/2$$

$$\text{Log}[z + 1] = \ln\,|\,z + 1| + i\,\text{Arg}[z+1]$$

$$\text{Im}[\text{Log}[z+1]] = \text{Arg}[z+1] = \text{Arg}[z]/2 = \Theta/2$$

(ii) Consider the periodic sawtooth function F(θ) whose graph is shown below. By substituting $z = e^{i\theta}$ in the logarithmic series (29), use the previous part to deduce the following Fourier series:

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$$F(\theta) = \sin\theta - \frac{\sin 2\theta}{2} + \frac{\sin 3\theta}{3} - \frac{\sin 4\theta}{4} + \frac{\sin 5\theta}{5} - \ldots.$$

$$(29)\ \text{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \ldots$$

$$\text{Im}\left(e^{in\theta}\right) = \text{Im}(\cos n\theta + i\sin n\theta) = \sin n\theta \qquad \text{(Euler)}$$

Substitute $e^{i\theta}$ for z in (29).

$$\text{Log}(1+z) = e^{i\theta} - \frac{e^{i2\theta}}{2} + \frac{e^{i3\theta}}{3} - \frac{e^{i4\theta}}{4} + \frac{e^{i5\theta}}{5} - \ldots$$

Apply Euler's equation, using Θ instead of θ, and take imaginary part.

$$F(\Theta) = \text{Im}(\text{Log}(1+z)) = \sin\Theta - \frac{\sin 2\Theta}{2} + \frac{\sin 3\Theta}{3} - \frac{\sin 4\Theta}{4} + \frac{\sin 5\Theta}{5} - \ldots$$

(iii) Check this Fourier series by directly evaluating the integrals (19).

From (1) and (2) F(Θ) = Θ/2.

$$b_n = \frac{1}{\pi}\int_0^{2\pi}(\Theta/2)\sin n\Theta\, d\Theta$$

$$=2\frac{1}{\pi}(1/2)\int_0^{\pi}\Theta\sin n\Theta\, d\Theta \qquad u = \Theta, \quad v = (-\cos n\Theta)/n,$$

$$du = d\Theta, \quad dv = \sin n\Theta\, d\Theta$$

$$= \frac{1}{\pi}\left(\Theta\,(-\cos n\Theta)/n \;\Big|_0^{\pi} - \int_0^{\pi}(-\cos n\Theta)/n\right) d\Theta$$

$$= \frac{1}{\pi}(-\Theta\,(\cos n\Theta)/n - (\sin n\Theta)/n^2)\,\Big|_0^{\pi}$$

$$= \frac{1}{\pi}\left[(-\pi\,(\cos n\pi)/n - (\sin n\pi)/n^2) - (0 - \sin 0/n^2)\right]$$

$$= \frac{1}{\pi}((-\pi\,(-1)^n/n - 0) = \frac{(-1)^{n-1}}{n}$$

$$b_1 = 1$$
$$b_2 = -\frac{1}{2}$$
$$b_3 = \frac{1}{3}$$

These coefficients agree with (29).

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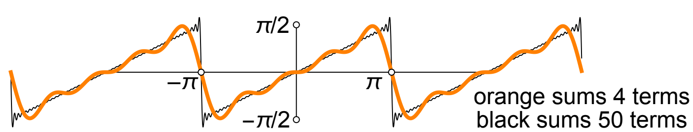

Figure 2. Smooth waves converge to jagged graph

(iv) Use a computer to draw graphs of the partial sums of the Fourier series. As you increase the number of terms, observe the magical convergence of this sum of smooth waves to the jagged graph above. If only Fourier could have seen this on screen, not just in his mind's eye!

$$F(\theta) = \sin\,\theta - \frac{\sin 2\,\theta}{2} + \frac{\sin 3\,\theta}{3} - \frac{\sin 4\,\theta}{4} + \frac{\sin 5\,\theta}{5} - \ldots$$

Rewrite this series as a sum, vary i to obtain number of terms, and plot (θ, $\Sigma(\theta$,i)) to plot Figure 2.

$$\sum_{n=1}^{i} (-1)^{n-1} \sin(\mathrm{n}\theta)/n$$

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