Needham, Chapter 2, Exercise 36, p. 119

G. Palmer, February 2, 2021, 12:10 am PDST

36. By considering Log(1+ix), where x is a real number lying between ± 1, deduce that

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \ldots$$

In what range does this value of $\tan^{-1}(x)$ occur? Give another derivation of the series using the idea in Ex. 31.

$$\text{Log}(1+ix) = \ln|1+ix| + i\text{Arg}(1+ix)$$

$$= \ln\sqrt{1+x^2} + i\tan^{-1}(x)$$

$$\tan^{-1}(x) = \text{Im}(\text{Log}(1+ix))$$

from equation (29), p. 101

$$\text{Log}(1+ix) = i\,x + \frac{x^2}{2} - \frac{i\,x^3}{3} - \frac{x^4}{4} + \frac{i\,x^5}{5} + \frac{x^6}{6} - \frac{i\,x^7}{7} - \frac{x^8}{8} + \ldots$$

$$\tan^{-1}(x) = \text{Im}(\text{Log}(1+ix))$$

$$= \text{Im}(i\,x + \frac{x^2}{2} - \frac{i\,x^3}{3} - \frac{x^4}{4} + \frac{i\,x^5}{5} + \frac{x^6}{6} - \frac{i\,x^7}{7} - \frac{x^8}{8} + \ldots)$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \qquad (1)$$

We check using Mathematica.

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Series[Log[1 + x I], {x, 0, 12}]
Im[Series[Log[1 + x I], {x, 0, 12}]] // Normal
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$$i\,x + \frac{x^2}{2} - \frac{i\,x^3}{3} - \frac{x^4}{4} + \frac{i\,x^5}{5} + \frac{x^6}{6} - \frac{i\,x^7}{7} - \frac{x^8}{8} + \frac{i\,x^9}{9} + \frac{x^{10}}{10} - \frac{i\,x^{11}}{11} - \frac{x^{12}}{12} + O[x]^{13}$$

$$\text{Im}\left[\frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} - \frac{x^8}{8} + \frac{x^{10}}{10} - \frac{x^{12}}{12}\right] + \text{Re}[x] - \frac{\text{Re}[x^3]}{3} + \frac{\text{Re}[x^5]}{5} - \frac{\text{Re}[x^7]}{7} + \frac{\text{Re}[x^9]}{9} - \frac{\text{Re}[x^{11}]}{11}$$

This agrees with (1) because the first term is zero.

In what range does this value of $\tan^{-1}(x)$ lie?

$$-\text{Pi}/4 < \tan^{-1}(x) < \text{Pi}/4$$

By the root test, the function has a radius of convergence at |x| = 1.

$\tan^{-1}(-1) = -\pi/4, \ \tan^{-1}(1) = \pi/4$

Give another derivation of the series [$\tan^{-1}(x) = Im[Log(1 + ix)]$] using the idea in Ex. 31.

You have to know that $\tan^{-1}(x) = \int \frac{1}{1+x^2}\, dx$.

$$\tan^{-1}(X) = \int_0^X \frac{1}{1+x^2}\, dx$$

$$\frac{1}{1+x^2} = (1+x^2)^{-1} = 1 - x^2 + \frac{-1\,(-1-1)}{2!}(x^2)^2 + \frac{-1\,(-1-1)\,(-1-2)}{3!}(x^2)^3 + \frac{-1\,(-1-1)\,(-1-2)\,(-1-3)}{4!}(x^2)^4 + \frac{-1\,(-1-1)\,(-1-2)\,(-1-3)\,(-1-4)}{5!}(x^2)^5 + \ldots$$

$$= 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \ldots$$

$$\int_0^X \frac{1}{1+x^2}\, dx = \int_0^X (1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \ldots)\, dx$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \ldots \Big|_0^X$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \ldots \quad \text{(after resubstituting x for X)}$$

This is the series we wanted.