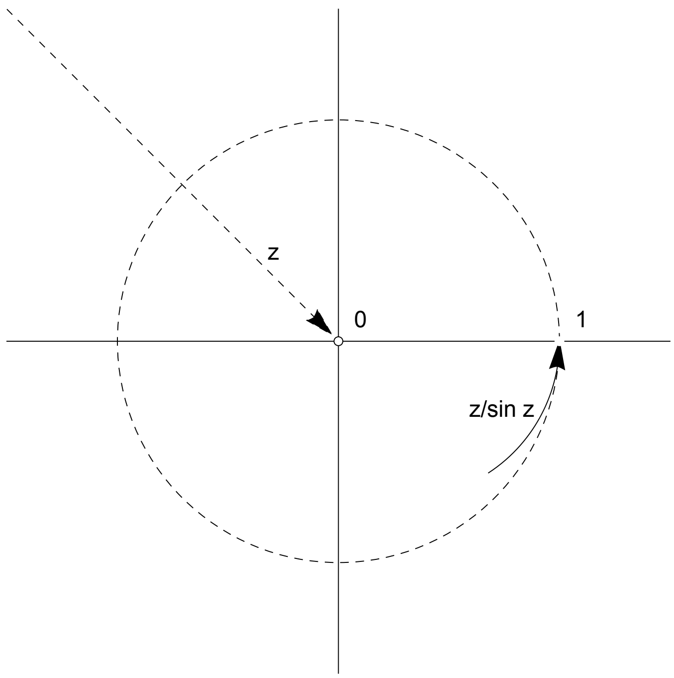
Needham, Chapter 2, Exercise 35, p. 119

G. Palmer, February 1, 2021, 1 pm PDST

35. What value does (z/sin z) approach as z approaches the origin? Use the series for sin z to find the first few terms of the origin-centered power series for (z/sin z). Check your answer using a computer. Where does this series converge?

Out[617]=



`Figure 1. Convergence of z/sin z as z approaches 0`

At the origin, z/sin z is undefined (0/0). As z approaches the origin, z/sin z approaches 1 (Figure 1).

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots$$

$$z/\sin z = \sum_{j=0}^{\infty} c_j z^j$$

$$z = \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ \ldots\right)\left(c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \ldots\right)$$

$$z = c_0 z \Longrightarrow c_0 = 1$$

$$0 = c_1 z^2 \Longrightarrow c_1 = 0$$

$$0 = c_2 z^3 - \frac{c_0 z^3}{3!} \Longrightarrow c_2 = \frac{c_0}{3!} = \frac{1}{3!}$$

$$0 = c_3 z^4 + \frac{c_1 z^4}{3!} \Longrightarrow c_3 = -\frac{c_1}{3!} = 0$$

$$0 = c_4 z^5 - \frac{c_2 z^5}{3!} + \frac{c_0 z^5}{5!} \Longrightarrow$$

$$c_4 = -\left(-\frac{c_2}{3!} + \frac{c_0}{5!}\right) = -\left(\frac{-5!\, c_2 + 3!\, c_0}{3! \times 5!}\right) = \frac{7}{360}$$

$$\Longrightarrow z/\sin z = 1 + \frac{z^2}{6} + \frac{7 z^4}{360} + \ldots$$

This is what Mathematica calculates.

In[436]:=
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Clear[z];

Series[z / Sin[z], {z, 0, 7}]
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Out[437]= $1 + \frac{z^2}{6} + \frac{7 z^4}{360} + \frac{31 z^6}{15120} + O[z]^8$

Where does this series converge?

There is a singularity at z = Pi and at every nPi. Then by (27) on p. 96, the radius of convergence is the distance to the nearest singularity. The series must converge where |z| < Pi.