Chapter 2, Exercise 34, p. 119

G. Palmer, January 17, 2021, 9:30 am PDST

34. Write down the origin-centred power series for (1-cos z). Use the Binomial Theorem to write down the power series (centred at Z = 0) for the principal branch of $\sqrt{1+Z}$*, then substitute Z = (1 - cos z). Hence show that if we choose the branch of* $\sqrt{\cos z}$ *that maps 0 to 1, then*

$\sqrt{\cos z} = 1 - \frac{z^2}{4} - \frac{z^4}{96} - \frac{19z^6}{5760} - \ldots$ [corrects mistake of sign on last term]

Verify this using a computer. Where does this series converge?

Using the origin–centered equation for the cos z on p. 85, we can write

$$1 - \cos z = \frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \ldots \qquad (1)$$

Now Vasco writes:

"The multifunction $\sqrt{1-Z}$, where $1-Z = Re^{i\theta}$, has a branch point at Z = 1. If we make a branch cut to infinity from Z = 1 in the direction of the negative real axis then the principal branch is $R^{1/2}\, e^{i\theta/2}$ and when Z = 0, the principal branch takes on the value $1^{1/2}\, e^{i0/2} = 1$, and the other branch takes on the value $1\, e^{i2\mathrm{Pi}/2} = -1$."

The power series obtained with the binomial theorem for $(1+Z)^n$ is given on p. 73. The series is centred at 0.

$$(1+Z)^n = 1 + nZ + \frac{n(n-1)}{2!} Z^2 + \frac{n(n-1)(n-2)}{3!} Z^3 + \cdots$$

The power series for the principal branch of $\sqrt{1+Z}$, in which n = 1/2.

$$(1+Z)^{1/2} = 1 + \tfrac{1}{2}Z + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!} Z^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!} Z^3 + \cdots$$

$$= 1 + \frac{Z}{2} - \frac{Z^2}{8} + \frac{Z^3}{16} - \cdots \qquad (2)$$

The power series for the principal branch of $\sqrt{1-Z}$. Note that

$$(1+(-Z))^{1/2} = \left(1+(-(1-\cos(z)))\right)^{1/2} = \pm\sqrt{\cos z}$$

The branch of $\sqrt{\cos z}$ which maps 0 to 1 is $+\sqrt{\cos z}$. Since the coefficients are identical to those for $\sqrt{1+Z}$ in (1), it is easy to find the result by writing minus signs in front of Z with odd exponents:

$$\sqrt{\cos z} = = 1 - \frac{Z}{2} - \frac{Z^2}{8} - \frac{Z^3}{16} - \cdots \qquad (3)$$

Substitute Z = 1 – cos z into (3).

$$\sqrt{\cos z} = 1 - \tfrac{1}{2}(1-\cos z) - \tfrac{1}{8}(1-\cos z)^2 - \tfrac{1}{16}(1-\cos z)^3 - \ldots$$

$$= 1 - \tfrac{1}{2} + \tfrac{1}{2}\cos z - \tfrac{1}{8}(1 - 2\cos z + \cos^2 z) - \tfrac{1}{16}(1 - 2\cos z + \cos^2 z)(1-\cos z)\cdots$$

$$= \frac{1}{2} + \frac{1}{2}\cos z - \frac{1}{8} + \frac{1}{4}\cos z - \frac{1}{8}\cos^2 z - \frac{1}{16}(1 - 2\cos z + \cos^2 z - \cos z + 2\cos^2 z - \cos^3 z)\cdots$$

$$\sqrt{\cos z} = \frac{5}{16} + \frac{15}{16}\cos z - \frac{5}{16}\cos^2 z + \frac{1}{16}\cos^3 z - \cdots \qquad (4)$$

Substitute $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$ on the right hand side of (4).

$$\sqrt{\cos z} = \frac{5}{16} + \frac{15}{16}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right) - \frac{5}{16}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right)^2 + \frac{1}{16}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right)^3 \cdots$$

The final result can be calculated (and verified) with a short computer script.

$$\sqrt{\cos z} = 1 - \frac{z^2}{4} - \frac{z^4}{96} - \frac{19 z^6}{5760} - \ldots.$$

This is what we wanted.

Verify this using a computer.

```
In[•]:= Clear[z];

(* power series for cos z *)
cosZ = 1 - z^2 / 2! + z^4 / 4! - z^6 / 6!;

(* power series for √cos z *)
sqrtCosZ[cos_] := Expand[5 / 16 + (15 / 16) cos - (5 / 16) cos^2 + (1 / 16) cos^3];

Print["√cos z = ", Take[sqrtCosZ[cosZ], 4]];

Print["Mathematica Series[Cos[z]^(1/2)]= ", Series[Cos[z]^(1/2), {z, 0, 6}]];
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$\sqrt{\cos z} = 1 - \frac{z^2}{4} - \frac{z^4}{96} - \frac{19 z^6}{5760}$

Mathematica Series[Cos[z]$^{1/2}$]= $1 - \frac{z^2}{4} - \frac{z^4}{96} - \frac{19 z^6}{5760} + O[z]^7$

This verifies our symbolic calculation, but Vasco (P.c. Forum) pointed out that we have to verify that the series produces the same answer as $\sqrt{\cos z}$. A first calculation on z = .5, 1, 1.5, 2, 3 using the series to z^8 suggests that the series diverges from $\sqrt{\cos z}$ in the fourth decimal where z = 1 but diverges at the first decimal at 1.5 and gets worse from there.

series results: {0.936543,0.735167,0.324734,-0.599854,-10.1838}
$\sqrt{\cos z}$ results: {0.936794,0.735053,0.265965,0.+0.645094 *i*,0.+0.994984 *i*}

Where does this series converge?

Since it appears that $\sqrt{\cos z}$ converges inside the unit disc, we can test this empirically on a series of points arranged on a diagonal through the origin at Arg(z) = π/4. In figure 1, the center of convergence appears to be 1. Reckoning from the centre of convergence, the first visible divergence of f(z) appears following z = ±(1 + i) and |z| = $\sqrt{2}$, the innermost points of z outside the unit (dashed) circle in Figure 1, below. The divergence remains small until after z = 1.2 + 1.2i. Then it accelerates rapidly. Hence, the radius of convergence appears to be between |1+i| ≃ 1.14 and |1.2+1.2i| ≃ 1.70. On p. 68, it says *If P(z) [polynomial] converges at* *z = a, then it will also converge everywhere inside the disc |z| = |a|.*

Out[ ]=

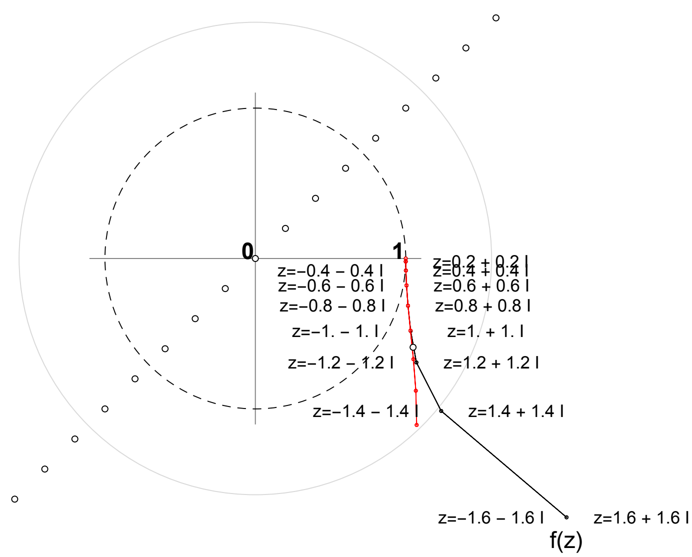

Figure 1. f(z) on points in diagonal $\sqrt{\cos z}$ (red) compared to series (black)

I have to back up now to (2), where $(1+(-Z))^{1/2} = \sqrt{\cos z}$ is expressed as a power series centred at 0. There is no singularity, so we can't use (15) on p. 75 to find the radius of convergence. But we can use (27) on p. 96: *If a complex function or a branch of a multifunction can be expressed as a power series, the radius of convergence is the distance to the nearest singularity or branch point* [u.l. added]. Our function appears to have a branch point where Z = 1, so Z = 1 − cos *z* implies that cos z = 0 and z = ±*π*/2. Then the radius of convergence is r = *π*/2, the radius of the light gray circle in Figure 1. The fifth dot *from 0* on the diagonal is just inside r. By consulting the line plots of f(z), we can see that the divergence occurs just beyond the fifth dot from the center of divergence. The point of divergence is marked with the white dot with black boundary. Note that both the black and red lines for f(z) merge their respective values for f(z) and f(−z).