Needham, Chapter 2, Exercise 33, p. 119
G. Palmer, January 29, 2020, 3:25 pm PDST

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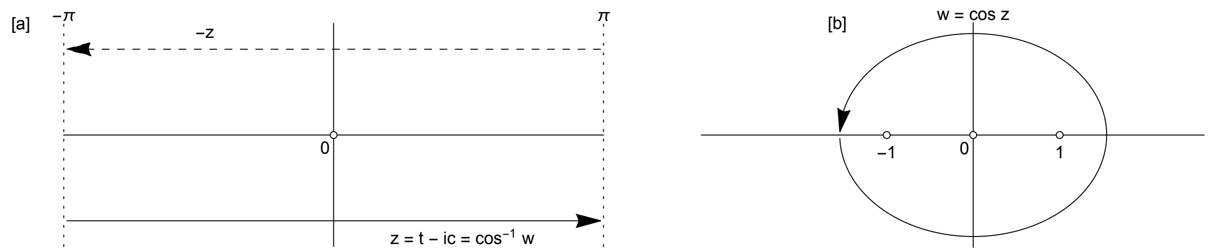

Figure 1. $\cos^{-1}(w)$ and $\cos(z)$

33. (i) Use [26] to discuss the branch points of the multifunction $\cos^{-1} z$.

Using [26], one can think of Figure 1[b] as the w plane, so w = cos z. Then one can change the problem statement to "discuss the branch points of the multifunction z = $\cos^{-1}$w." Applying the arccosine to w produces a family of straight lines in the z-plane. Referring strictly to the curve cos z shown in [26], one can guess that branch points occur at the two foci. Then we would expect that the inverse function would produce a family of lines parallel to lines in Figure 1[a]. One can see that these lines do not return to their starting place after a single loop of w through an angle of 2π, so it appears that the foci are actually branch points. But how can we be more certain. As a start, we can let w traverse a closed loop around one of the foci. If $z = \cos^{-1} w$ does not return to its starting point, then the focus is a branch point. We can also let w traverse a closed loop that contains zero. If z returns to its starting point, then we know that 0 is not a branch point.

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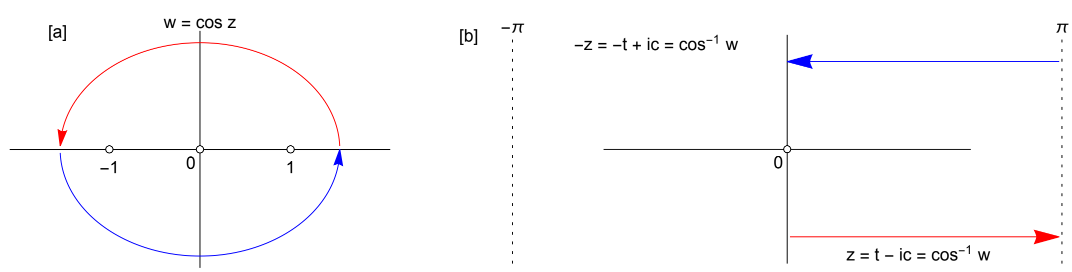

Figure 2. $\cos(z)$ to $\cos^{-1}(w)$

Causing z to transit from (π + i c) to ($-\pi$ + i c) in the upper half plane (UHP) produces the directed ellipse seen in Figure 1[b]. The same directed ellipse is produced when z is caused to transit from ($-\pi$ – ic) to (π – i c) in the lower half plane (LHP). If we now reverse the order of [a] and [b] and cause w to traverse the ellipse in Figure 2 [a] once in an anticlockwise direction beginning at $\mathrm{Arg}(w) = -\pi$, the result shown in [b] is a left-directed line in the UHP from (π + i c) to i c followed by a right-directed line in the z-plane in the LHP from (-ic) to (π – ic) showing that what began as π + ic terminates at π – ic. Hence, neither of the directed arrows shown in Figure 1[a] return to their starting point after a single complete loop of w that includes the foci and the origin. The blue arrow in [b] is generated from the blue arrow in [z] and the red in [b] is generated from the red in [a].

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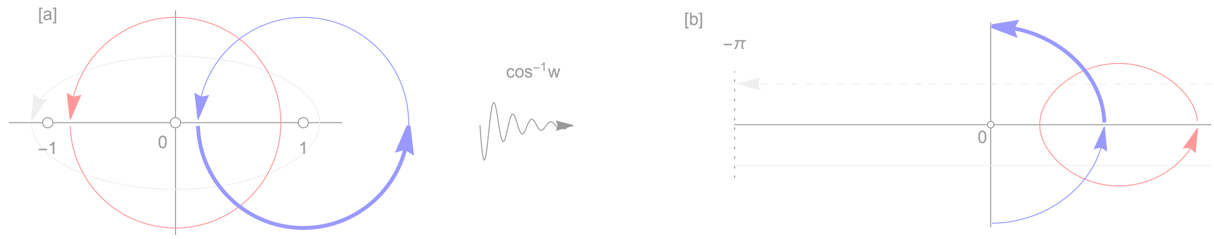

Figure 3. $\text{Cos}^{-1}(w)$ on circles centred at origin and 1

What if we apply the test to one of the origin-centred circles on the RHS of (26)? Neither of these circles include a focus point, so we can determine whether the origin is a branch point. Take the circle $\frac{1}{2}e^{iz}$ (red circle in Figure 3[a]). Apply $z = \cos^{-1}\left(\frac{1}{2}e^{iz}\right)$, beginning at $\text{Arg}\left(\frac{1}{2}e^{iz}\right) = -\pi$ and terminating at $\text{Arg}\left(\frac{1}{2}e^{iz}\right) = \pi$. Then $\text{Arg}\left(\frac{1}{2}e^{iz}\right) = -\pi$ implies that $t = -\pi$, and $\text{Arg}\left(\frac{1}{2}e^{iz}\right) = \pi$ implies that $t = \pi$, because (26) shows that t is the angle of w traversing a circle in the w-plane, and because

$$\frac{1}{2}e^{iz} = \frac{1}{2}e^{i(t-ic)} = \frac{1}{2}e^{c}e^{it}\text{, so } \text{Arg}\left(\frac{1}{2}e^{c}e^{it}\right) = t$$

$$\text{starting point:}\quad \cos^{-1}\left(\frac{1}{2}e^{c}e^{-i\pi}\right) = \cos^{-1}(-e^{c}/2)$$

$$\text{termination:}\quad \cos^{-1}\left(\frac{1}{2}e^{c}e^{i\pi}\right) = \cos^{-1}(-e^{c}/2)$$

Since the arguments of the arccosine are equal, it appears that z returns to its starting point. We can see this in the red ellipse in Figure 3 [b]. So we can rule out the origin as a branch point. In the same way, we can rule out i and $-i$. But what about a focal point such as 1?

$$\text{starting point:}\quad \cos^{-1}\left(\frac{1}{2}e^{iz}+1\right) = \cos^{-1}\left(\frac{1}{2}e^{-i\pi}+1\right) = \cos^{-1}(1-e^{c}/2)$$

$$\text{termination:}\quad \cos^{-1}\left(\frac{1}{2}e^{iz}+1\right) = \cos^{-1}\left(\frac{1}{2}e^{i\pi}+1\right) = \cos^{-1}(1-e^{c}/2)$$

It appears we have again ruled out a focal point as a branch point, because the arguments of the arccosine are equal. Furthermore, the normal and thick blue arrows in Figure 3[b] show that z does return to its starting point at . But there is a discontinuity at the y-axis which indicates that the two blue arrows derive from different branches of the arccosine. Then it is really the case that neither of the two arrows in [b] returns to its starting point. In fact, $\cos^{-1}(w)$ applied to a circle that contains the focal point 1 behaves similarly to $\sqrt[3]{z}$ in [29]. For each full traversal of w round this point, the traversal of z round the origin resembles that of $\sqrt{z}$, except that there are discontinuities in the half-circle result of $\cos^{-1}(w)$. We conclude that focal point 1 is a branch point. Why did I fail to make this determination using the algebra?

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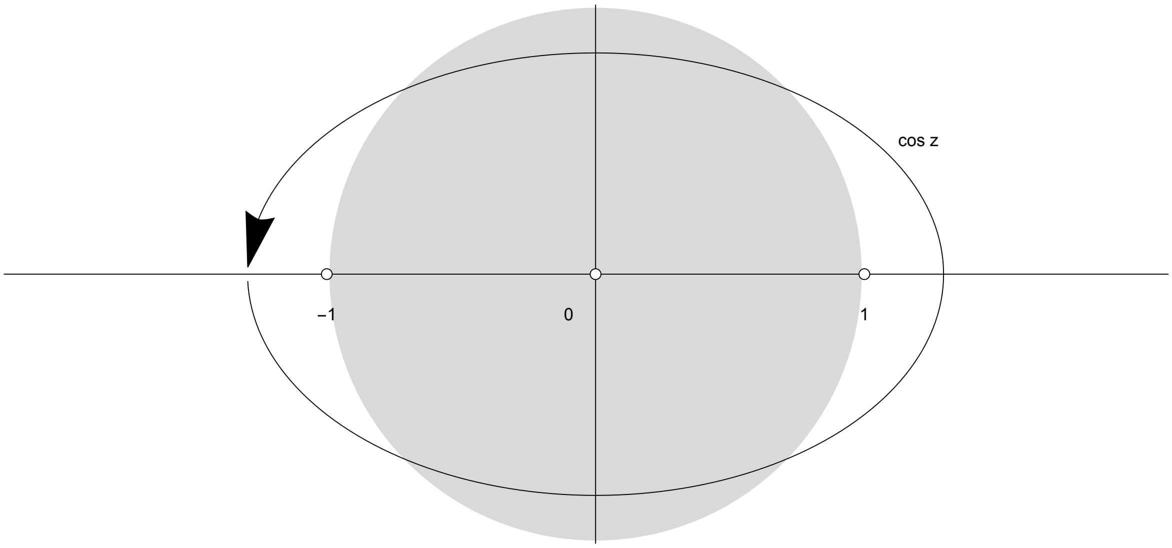

Figure 4. Disc of convergence for $\cos^{-1}(w)$ power series

Further considerations:

The power series for $\cos^{-1} x$ is

$$\cos^{-1} x = \pi/2 - x - \frac{x^3}{2\cdot 3} - \frac{3\,x^5}{2\cdot 4\cdot 5} - \frac{3\cdot 5\,x^7}{2\cdot 4\cdot 6\cdot 7} - \cdots$$

We know (from a later chapter) that this can be extended to the complex plane, so

$$\cos^{-1} z = \pi/2 - z - \frac{z^3}{2\cdot 3} - \frac{3\,z^5}{2\cdot 4\cdot 5} - \frac{3\cdot 5\,z^7}{2\cdot 4\cdot 6\cdot 7} - \cdots$$

Since $\cos^{-1} x$ is known to converge at $|x| < 1$, $\cos^{-1} z$ converges at $|z| < 1$. Then the four points at $e^{\pm i n\pi/2}$ are likely candidates for branch points. The two foci of the ellipse, 1 and -1, seem most likely to be branch points.

Plot a circle of radius 1 centred at the origin. Also plot the ellipse in (26). Then it becomes evident that the only portion of this particular ellipse that is in the principal branch of $\cos^{-1} z$ is that lying within the gray circle.

(ii) Rewrite the equation $w = \cos z$ as a quadratic in e^{iz}. By solving this equation, deduce that $\cos^{-1}(z) = -i \log[z + \sqrt{z^2 - 1}\,]$. [Why do we not need to bother to write ± in front of the square root?]

Vasco says: *Rewrite the equation w = cos z as a quadratic in* e^{iz}*. By solving this equation, deduce that* $\cos^{-1} w = -i \log[w + \sqrt{w^2-1}]$*. This would mean that we should also rewrite part (iii) as: Show that as w travels along a loop that goes once round either 1 or –1 (but not both), the value of* $[w + \sqrt{w^2-1}]$ *changes to* $1/[w + \sqrt{w^2-1}]$.

We have a solution that doesn't use quadratic of e^{iz}. Euler's equation is sufficient.

$w = \cos z$ (given)

$z = \cos^{-1}(\cos z) = \cos^{-1} w$

From Figure [26],

$z = -i\, iz = -i \operatorname{Log}[2\, \frac{1}{2} e^{iz}] = -i \operatorname{Log}[e^{iz}]$

$= -i \operatorname{Log}[\cos z + i \sin z]$ (by Euler's equation)

$= -i \operatorname{Log}[\cos z + \sqrt{-\sin^2 z}\,]$

$= -i \operatorname{Log}[\cos z + \sqrt{(1 - \sin^2 z) - 1}\,]$

$= -i \operatorname{Log}[\cos z + \sqrt{\cos^2 z - 1}\,]$

$= -i \operatorname{Log}[w + \sqrt{w^2-1}\,]$

Hence, $\cos^{-1} w = -i \operatorname{Log}[w + \sqrt{w^2-1}\,]$.

The solution with the quadratic:

$w = \cos z = \frac{e^{iz} + e^{-iz}}{2}$ and $z = \cos^{-1} w$

$2w = e^{iz} + e^{-iz}$

$2\, w e^{iz} = e^{2\, iz} + 1$ multiply through by e^{iz}

$e^{2\, iz} - 2\, w e^{iz} + 1 = 0$ rearrange

$e^{iz} = \frac{2\, w + \sqrt{4\, w^2 - 4}}{2} = w + \sqrt{w^2-1}$ solve quadratic in e^{iz}

$i\, z = \operatorname{Log}(w + \sqrt{w^2-1}\,)$ take Log of both sides

$\cos^{-1} w = -i \operatorname{Log}(w + \sqrt{w^2-1}\,)$ multiply through by –i; $z = \cos^{-1} w$

[Why do we not need to bother to write ± in front of the square root?]

Since we have $\cos^{-1} w$ in Log form, we can rewrite the multifunction as

$\cos^{-1} w = -i \log(|w + \sqrt{w^2-1}\,|) = -i\, [\ln|w + \sqrt{w^2-1}\,| + i\operatorname{Arg}(w + \sqrt{w^2-1}\,) \pm 2n\pi i]$

From this one can deduce that whatever the value of $\arg(w + \sqrt{w^2-1}\,)$, $\cos^{-1} w$ has multiple values.

We find empirically that $-i \operatorname{Log}(w + \sqrt{w^2-1}\,) = i \operatorname{Log}(w - \sqrt{w^2-1}\,)$. This is a difference that matters, but so long as we are aware of the difference, we know that $\cos^{-1} w$ has two values (sub-branches?) of opposite sign within each branch, so perhaps this is why it is not necessary to write ± on the root. It should follow that $\left(w - \sqrt{w^2-1}\right) = 1\big/\left(w + \sqrt{w^2-1}\right)$.

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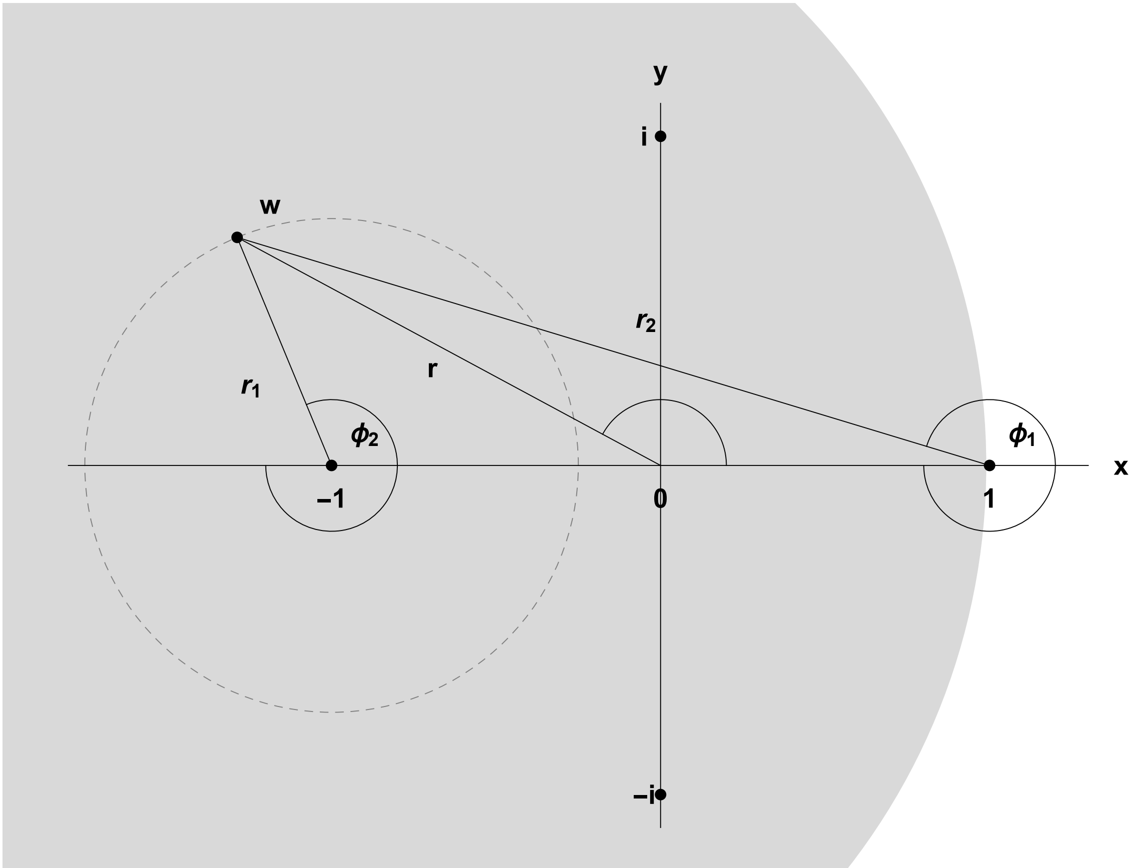

Figure 2. Branch points and angles of [w + $\sqrt{(w-1)(w+1)}$]
w loops around branch point –1

(iii) Show that as w travels along a loop that goes once round either 1 or –1 (but not both), the value of [w + $\sqrt{w^2-1}$ *] changes to 1/[w +* $\sqrt{w^2-1}$ *].*

Let $w = re^{i\phi} - 1$, where r may vary, but r < 2 (see partial gray disc in Figure 2), and where $\phi = \phi_2$ from Figure 2.

Then $\sqrt{w^2-1} = \sqrt{r^2 e^{i2\phi} - 2re^{i\phi}}$.

Since the principal branch loop begins at $\phi = -\pi$ and terminates at π, we can write the starting point in the z-plane as

$$z = w + \sqrt{w^2-1} = re^{i\phi} + \sqrt{r^2 e^{i2\phi} - 2re^{i\phi}} = re^{-i\pi} + \sqrt{r^2 e^{-i2\pi} - 2re^{-i\pi}}$$

After one trip of w around the loop in the anticlockwise direction, we can write

$$z = re^{i(-\pi+2\pi)} + \sqrt{r^2 e^{i2(-\pi+2\pi)} - 2re^{i(-\pi+2\pi)}} = re^{i\pi} + \sqrt{r^2 e^{i2\pi} - 2re^{i\pi}}$$

Then it should be the case that $re^{i\pi} + \sqrt{r^2 e^{i2\pi} - 2re^{i\pi}} = 1/(re^{-i\pi} + \sqrt{r^2 e^{-i2\pi} - 2re^{-i\pi}})$. I have not been able to prove this.

If we use a loop of radius 1, so that r = 1, then the RHS reduces to $e^{-i\pi} \to e^{i\pi}$ and since we know that $e^{i\pi} = 1/e^{-i\pi}$, it follows that, where r = 1, after one trip around the loop, $w + \sqrt{w^2-1}$ becomes $1/[w + \sqrt{w^2-1}]$. I do not see how to generalize it to a different radius or a variable radius.

Another approach:

Begin with the idea that Log(z) = – Log(1/z) and consider Figure 3. We might expect the traversal of w on the half of the loop in the LHP of [z] to plot z's on the left hand side in the LHP of Figure 3[b]. In fact, the z's are plotted on the right hand side of the UHP. Then we might expect the continuation of the traversal of w round the ellipse in the UHP of [a] to plot z's on left hand side of the UHP, but in fact they plot on the right hand side of the LHP. $\mathrm{Cos}^{-1}\, w$ has no points with Re(z) < 0, when Arg(w) is confined to the principle values. What has happened? It appears that all instances where $\mathrm{Re}(-\,\mathrm{i}\,\mathrm{Log}(w + \sqrt{w^2-1})) < 0$ have changed sign, which indicates that $\mathrm{Log}(w + \sqrt{w^2-1})$ has changed sign, which indicates that $w + \sqrt{w^2-1}$ has become $1/(w + \sqrt{w^2-1})$.

(iv) Use the previous part to show that the discussion in part (ii) is in accord with the discussion in part (i).

The discussion in part (i) asserts that $\cos^{-1} w$ has at least one branch point because z in Figure 1[a] does not return to its original point after one loop of w in Figure 1[b]. It appears that $\cos^{-1} w$ has an infinite number of branches based using the foci as branch points and letting w loop around the ellipse. The discussion in part (ii) centred on the lack of need to write the root with positive and negative sign. I argued there that $w - \sqrt{w^2-1}$ would be equivalent to $1/[w + \sqrt{w^2-1}]$. In part (iii), I tried to show that on one trip around a single closed loop, $w + \sqrt{w^2-1}$ becomes $1/[w + \sqrt{w^2-1}]$. I showed it with a special case, but I was unable to generalize it. But one can argue that part (iii) is consistent with my argument in part (ii) if $\sqrt{w^2 - 1}$ becomes $-\sqrt{w^2 - 1}$ after one loop. If we set these equal, we find that $e^{\mathrm{i}4\pi} = 1$, which is true.

$$\sqrt{r^2\, e^{i2\pi} - 1} = -\sqrt{r^2\, e^{-i2\pi} - 1}$$

$$r^2\, e^{i2\pi} - 1 = r^2\, e^{-i2\pi} - 1$$

$$r^2\, e^{i2\pi} = r^2\, e^{-i2\pi}$$

$$e^{i2\pi} = e^{-i2\pi}$$

$$e^{\mathrm{i}4\pi} = 1$$