Needham, Chapter 2, Exercise 32, p. 119
G. Palmer, December 20, 2020, 4 pm PDST

32 Here is another approach to the logarithmic power series. As before, let L(z) = Log(1+z). Since L(0) = 0, the power series for L(z) must be of the form $L(z) = az + bz^2 + cz^3 + dz^4 + \cdots$. *Substitute this into the equation*

$$1 + z = e^L = 1 + L + \frac{1}{2!}L^2 + \frac{1}{3!}L^3 + \frac{1}{4!}L^4 \cdots,$$

then find a, b, c, and d by equating powers of z.

$$e^L = e^{az + b z^2 + c z^3 + d z^4 + \cdots}$$

Note: The Mathematica editor misfunctions when one attempts to enter a superscripted power. It spaces the power incorrectly and disrupts page layout. I have therefore replaced the faulty superscripts with “^n”.

On the RHS, sum of

1 = 1
L = $(az + bz^2 + cz^3 + dz^4 + \cdots)$
$\frac{1}{2!}$L^2 = $(az + bz^2 + cz^3 + dz^4 + \cdots)$^2
$\frac{1}{3!}$L^3 = $\frac{1}{3!}$ $(az + bz^2 + cz^3 + dz^4 + \cdots)$^3
$\frac{1}{4!}$L^4 = $\frac{1}{4!}$ $(az + bz^2 + cz^3 + dz^4 + \cdots)$^4

equals sum of

1
az + bz^2 + cz^3 + dz^4 + ⋯
$\frac{1}{2!}$ (a^2 z^2 + 2 a bz^3 + (b^2 + 2 a c) z^4 ...)
$\frac{1}{3!}$ (a^3 z^3 + 3 a^2 bz^4 ⋯)
$\frac{1}{4!}$ a^4 z^4 ⋯

Then expand the sum

(1+z) = 1 + az + (b + $\frac{1}{2}$a^2) z^2 + (c + a b + $\frac{1}{6}$ a^3) z^3 + (d + $\frac{1}{2}$b^2 + a c + $\frac{1}{2}$a^2 b + $\frac{1}{24}$a^4) z^4 ⋯

0 = (a – 1) z + (b + $\frac{1}{2}$a^2) z^2 + (c + a b + $\frac{1}{6}$ a^3) z^3 + (d + $\frac{1}{2}$b^2 + a c – $\frac{1}{2}$a^2 b + $\frac{1}{24}$a^4) z^4 ⋯

The powers on the LHS are now all 0. Write four equations for a, b, c, and d, equating powers of z.

$a - 1 = 0$
$a = 1$

$b + \frac{1}{2}a^2 = 0$
$b = -\frac{1}{2}$

$c + a\,b + \frac{1}{6}a^3 = 0$
$c - \frac{1}{2} + \frac{1}{6} = 0$
$c = \frac{1}{3}$

$d = -\frac{1}{2}b^2 - a\,c - \frac{1}{2}a^2\,b - \frac{1}{24}a^4$
$= -\frac{1}{8} - \frac{1}{3} + \frac{1}{4} - \frac{1}{24}$
$= -\frac{3}{24} - \frac{8}{24} + \frac{6}{24} - \frac{1}{24} = -\frac{6}{24}$
$= -\frac{1}{4}$

The result is $a = 1$, $b = (-\frac{1}{2})$, $c = \frac{1}{3}$, and $d = -\frac{1}{4}$.

$$L(z) = az + bz^2 + cz^3 + dz^4 = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 \cdots$$

This agrees with the result in (29) and by extension from real to complex numbers, our results in Exercise 31.