Needham, Chapter 2, Exercise 31, p. 119
G. Palmer, December 14, 2020, 9:40 am PDST

30 In the case of a real variable, the logarithmic power series was originally discovered [see next exercise] as follows. First check that ln(1+X) can be written as $\int_0^X [1/(1+x)\, dx$*, and then expand [1/(1+x)] as a power series in x. Finally, integrate your series term by term.*

$$\int_0^X [1/(1+x)\, dx = ln(1+x)\, |_0^X$$

$$= [\ln(1+X) - \ln(1)] = \ln(1+X)$$

This shows that ln(1+X) can be written as $\int_0^X [1/(1+x)\, dx$.

The formula calls for an expansion centred at –1. The largest disk of convergence we can have has radius 1. From p. 64 (3), we have, if and only if $(-1 < -x < 1)$,

$$1/(1+x) = 1/(1-(-x)) = \sum_{j=0}^{\infty} (-x)^j$$

$$= 1 + (-x)^1 + (-x)^2 + (-x)^3 + \ldots$$

$$= 1 - x + x^2 - x^3 + \ldots.$$

Integrate term by term.

$$ln(1+X) = \int_0^X [1/(1+x)\, dx = \int_0^X \left[1 - x + x^2 - x^3 + \ldots\right)\, dx$$

$$= \int_0^X 1\, dx - \int_0^X x\, dx + \int_0^X x^2\, dx - \int_0^X x^2\, dx$$

$$= X - X^2/2 + X^3/3 - X^4/4 + \ldots$$

which is equivalent to

$$\ln(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \ldots$$