Needham, Chapter 2, Exercise 30, p. 119
G. Palmer, December 10, 2020, 9:05 am PDST

30 Show that all the values of i^i are real! Are there any other points z such that z^i is real?

i^i appears to be an instance of k^z, where k is complex. From Ex. 29, $k^z = e^{l_n z}$. Then, where n = ±1, ±2, ±3, ...,

$$\log i^i = [\ln|i| + i\mathrm{Arg}(i) + 2n\pi]\, i$$

$$= [\ln 1 + i\,\pi/2 + 2n\pi i\,]\, i$$

$$= 0 - \pi/2 - 2n\pi$$

$$i^i = e^{-(\pi/2 + 2\,n\pi)}$$

This shows that all the branches of i^i are real, because e to a real power is real and each value of n spawns a new branch.

Out[ ]=

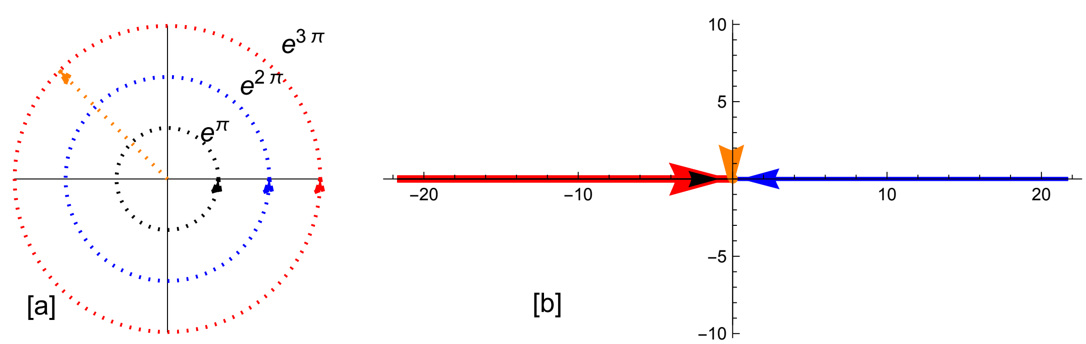

Figure 1. z^i on z.

Are there any other points z such that z^i is real?

Since z and i are both complex, we can reverse the positions of z and k, so that z is the base and k = i is the power. Then write

$$z^i = e^{i\, l_n} = e^{i\,[\ln|z| + i\,\mathrm{Arg}(z) + 2\,n\pi i]}$$

$$= e^{[i\,\ln|z| - (\mathrm{Arg}(z) + 2\,n\pi i)]}$$

$$= e^{-(\mathrm{Arg}(z) + 2\,n\pi i)} e^{i\,\ln|z|}$$

$$= e^{-(\mathrm{Arg}(z) + 2\,n\pi)}(\cos(\ln|z|) + i\,\sin(\ln|z|))$$

Since we want z^i to be a real number, it must be the case that sin(ln|z|) = 0. Then ln|z| = mπ, where m is an integer. Since x = $e^{\ln(x)}$,

$$|z| = e^{\ln|z|} = e^{m\pi}$$

Then we conclude as Vasco did that the points of z lie on a series of concentric circles with radii $e^{m\pi}$. Then returning to part (i)

$$i^i = \left(|z|\, e^{i \arg(z)}\right)^i = \left(e^{m\pi} e^{i \arg(z)}\right)^i$$

$$= \left(e^{0\cdot\pi} e^{i \arg(z)}\right)^i = 1 \cdot e^{i\,[i\,(\pi/2+2\,n\pi)]}$$

$$= e^{-\,(\pi/2+2\,n\pi)}$$

where n = ±1, ±2, ...

which accords with the answer to part (i).

Figure 1. Shows that the three concentric circles in [a] plot under z^i as three arrows on the real line in [b]. The dotted ray in [a] apparently plots as a tight cluster of points surrounding the origin in [b]. Only the orange arrowhead and the cluster are visible. None of the transformed ray points in [b] are actually on the origin or the real line, except where the ray intersects the circles. The concentric circles and the ray in [a] are actually on a log scale, as $e^{m\pi}$ rapidly gets so large that it is hard to put all the values on the unmodified scale. But in [b] all points of z^i are plotted on the correct values of z.