Needham, Chapter 2, Exercise 3, p. 111.
G. Palmer, May 20, 2020, 8:10 am PST

3. Consider the family of complex mappings

$$z \longrightarrow M_a(z) = \frac{z-a}{\overline{a}\,z-1} \qquad \textit{(a constant) [but still complex]}$$

Do the following problems algebraically; in the next chapter we will provide geometric explanations.

(i) Show that $M_a[M_a[(z)] = z$. In other words M_a is self-inverse.

$$\frac{\frac{z-a}{\overline{a}z-1}-a}{\overline{a}\,\frac{z-a}{\overline{a}z-1}-1} = \frac{\frac{z-a}{\overline{a}z-1}-a\,\frac{\overline{a}z-1}{\overline{a}z-1}}{\overline{a}\,\frac{z-a}{\overline{a}z-1}-\frac{\overline{a}z-1}{\overline{a}z-1}}$$, by substitution of $\frac{z-a}{\overline{a}z-1}$ for z

$$= \frac{z-a-|a|^2 z+a}{\overline{a}\,z-|a|^2-\overline{a}\,z+1}$$, by multiplying numerator and denominator by $\overline{a}\,z-1$

$$= \frac{z\left(1-|a|^2\right)}{1-|a|^2} = \text{z}$$

(ii) Show that $M_a(z)$ maps the unit circle to itself.

We want to show that $|M_a(z)| = 1$.

$$M_a(z) = w = \frac{z-a}{\overline{a}\,z-1}$$

$$|\,w\,|^2 = \left|\,\frac{z-a}{\overline{a}\,z-1}\,\right|^2$$

$$= \frac{|z-a|^2}{|\overline{a}\,z-1|^2}$$

$$= \frac{(z-a)\,\overline{z-a}}{(\overline{a}\,z-1)\,\overline{\overline{a}\,z-1}} \quad \text{(because } |\,z\,|^2 = z\overline{z}\text{)}$$

$$= \frac{|z|^2-\overline{a}\,z-a\,\overline{z}+|a|^2}{\overline{a}\,\mathrm{az}\,\overline{z}-\overline{a}\,z-a\,\overline{z}+1}$$

$$= \frac{|z|^2-\overline{a}\,z-a\,\overline{z}+|a|^2}{|a|^2|z|^2-\overline{a}\,z-a\,\overline{z}+1}$$

$$= \frac{1-\overline{a}\,z-a\,\overline{z}+|a|^2}{|a|^2-\overline{a}\,z-a\,\overline{z}+1} \text{ (if z is on the unit circle, |z| = 1 = } |\,z\,|^2\text{)}$$

$$= \frac{|a|^2-\overline{a}\,z-a\,\overline{z}+1}{|a|^2-\overline{a}\,z-a\,\overline{z}+1} = 1$$

Since $|w|^2 = 1$, then $|w| = 1$, so $M_a(z)$ maps the unit circle to itself.

What have we done? We have obtained an equation $|w| = 1$ and shown that the equation depends on $|z| = 1$. We obtained 1 by squaring the absolutes and substituting for $|z|^2$.

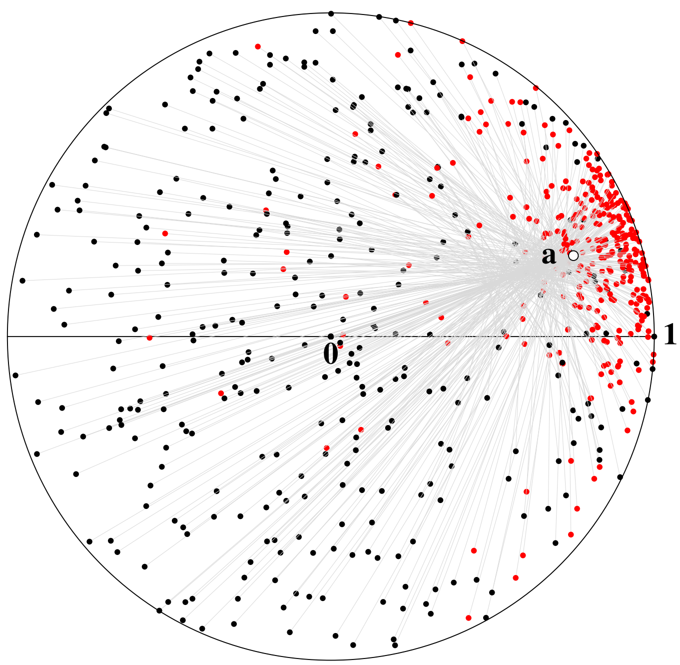

Figure 1. Mapping the unit disc to itself
$M_a = (z-a)/(\bar{a}z-1)$

(iii) Show that if a lies inside the unit disc then $M_a(z)$ maps the unit disk to itself. Hint: Use $|q|^2 = q\,\bar{q}$ to verify that

$$|\bar{a}\,z - 1|^2 - |z - a|^2 = (1 - |a|^2)\,(1 - |z|^2).$$

$$|\bar{a}\,z - 1|^2 - |z - a|^2 = (\bar{a}\,z - 1)\,\overline{\bar{a}\,z - 1} - (z-a)\,\overline{z-a}$$

$$= (\bar{a}\,z - 1)\,(a\,\bar{z} - 1) - (z-a)\,\overline{z-a}$$

$$= \left(|a|^2\,|z|^2 - \bar{a}\,z - a\bar{z} + 1\right) - \left(|z|^2 - \bar{a}\,z - a\,\bar{z} + |a|^2\right)$$

$$= \left(|a|^2\,|z|^2 + 1\right) - \left(|z|^2 + |a|^2\right)$$

$$= 1 - |z|^2 - |a|^2 + |a|^2\,|z|^2 = (1 - |a|^2)\,(1 - |z|^2)$$

We have verified the equation, but we still have to show that $M_a(z)$ maps the unit disc to itself. If a is inside the unit disc, $1 - |a|^2 > 0$. If z inside or on the unit disc, $1 - |z|^2 \geq 0$. Then it follows that

$$(1 - |a|^2)(1 - |z|^2) \geq 0$$

Then since

$$|\overline{a}\, z - 1|^2 - |z - a|^2 = (1 - |a|^2)(1 - |z|^2)$$

it follows that

$$|\overline{a}\, z - 1|^2 - |z - a|^2 \geq 0.$$

$$1 - \frac{|z-a|^2}{|\overline{a}\, z-1|^2} \geq 0 \implies 1 - |w|^2 \geq 0, \text{ by dividing through by } |\overline{a}\, z - 1|^2$$

$$|w|^2 \leq 1$$

so the closed unit disc is mapped to itself. In Figure 1, we demonstrate this using black for 400 random z points and red for $M_a[z]$ points. It appears that given enough points, M_a maps to the whole disk, but that the density of M_a points is highest in a neighborhood between **a** and the disc border. It has the appearance of an inversion in **a**.