Needham, Chapter 2, Exercise 29, p. 118.
G. Palmer, December 6, 2020, 4:20 pm PDST

29 In this exercise you will see that the multifunction k^z is quite different in character from all the other multifunctions we have discussed. For integer values of n, define $l_n \equiv [\mathrm{Log}(k) + 2n\pi i]$.

(i) Show that the "branches" of k^z are $e^{l_n z}$.

In VII 3, General Powers (pp. 101-102), Needham treats "k", as either an integer or a complex number. Then one can write

$$\mathrm{Log}(k) = \ln|k| + i\,\mathrm{Arg}(k)$$

$$l_n = [\ln|k| + i\,\mathrm{Arg}(k) + 2n\pi i] = \log(k)$$

$$\log(k^z) = z\log(k) = z\,l_n$$

On p. 101, equation (30), we have $z^k = e^{k\log(z)}$. Since k may be a complex number, we can write the inverse of (30) with respect to k and z:

$$k^z = e^{z\log(k)} = e^{z\,l_n} = e^{l_n z}$$

Rewrite $e^{l_n z}$ as

$$e^{[\ln|k|+i\,\mathrm{Arg}(k)]\,z}\,e^{i2n\pi z} = e^{i2n\pi z}e^{z\,\mathrm{Log}(k)} = (e^{i2n\pi}[e^{\mathrm{Log}(k)}])^z$$

Since $e^{z\,l_n}$ has a period of 2πi, we have shown that the "branches" of k^z are $e^{l_n z}$.

In Figure 1, below, we show a dotted loop beginning where p = -.7 + .7i and a sold loop beginning at k^p showing that when z returns to p, k^z returns to k^p. The black on red dot is explained in part (ii).

Out[1124]=

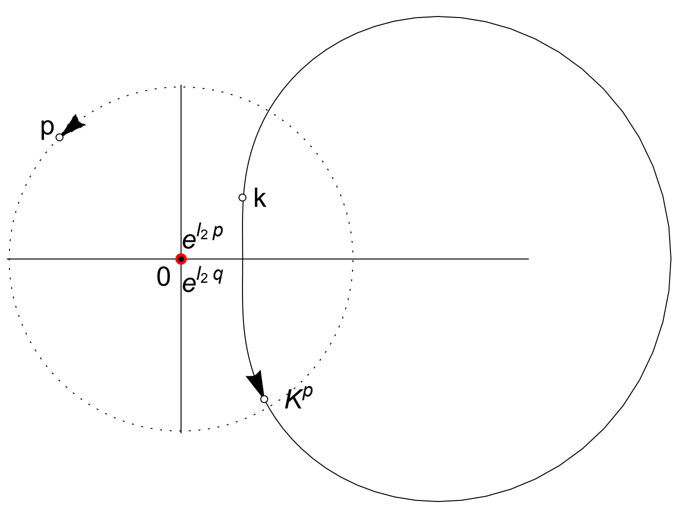

Figure 1. k^z, z traverses closed loop

(ii) Suppose that z travels along an arbitrary loop, beginning and ending at z = p. If we initially choose the value $e^{l_2 p}$ for k^z, then what value of k^z do we arrive at when z returns to p? Deduce that k^z has no branch

points.

Let $p = r\, e^{i\theta}$. After one traversal of a closed loop, this becomes $q = r\, e^{i(\theta+2\pi)} = re^{i\theta}e^{i2\pi} = pe^{i2\pi} = p$. Figure one shows that $e^{l_2 p} = e^{l_2 q}$. No traversal is shown for this part of the problem.

$$k^q = e^{l_2 q} = e^{l_2 p} = k^p$$

This shows that k^z returns to k^p after one trip around a closed loop, so it must return to k^p after every such trip. This is indicated by the black on red point at the origin. When z is zero, $k^z = e^{l_2 z} = 1$. When z is a negative integer, $e^{l_2 z} = 1/e^{n\, l_2}$, which would approach 0 as $n \longrightarrow \infty$, but there is no zero point or singularity. Since k^z always returns to its starting place after one traversal, there must be no branch points.

Since we cannot change one value of k^z into another by traveling around a loop, we should view its "branches" $\{\ldots, e^{l_{-1} z}, e^{l_0 z}, e^{l_1 z}, \ldots\}$ as an infinite set of <u>completely unrelated</u> single-valued functions.