Needham, Chapter 2, Exercise 27, p. 118.
Gary Palmer, December 4, 2020, 7:30 am PDST

27 Until Euler cleared up the whole mess, the complex logarithm was a source of tremendous confusion. For example, show that log(z) and log(-z) have no common values, then consider the following argument of John Bernoulli:

$$log\left[(-z)^2\right] = log[z^2] \Longrightarrow log(-z) + log(-z) = log(z) + log(z)$$

$$\Longrightarrow 2\ log(-z) = 2\ log(z)$$

$$\Longrightarrow log(-z) = log(z)$$

What is wrong with this argument?

From p. 98, we have

$$\log(z) = \ln|z| + i \arg(z)$$

Let $z = r\,e^{i\theta}$, where $r = |z|$. Then $-z = r\,e^{i(\theta+\pi)}$ and

$$\log(z) = \ln r + i \arg(re^{i\theta}) = \ln r + i\,(\theta + 2n\pi), \qquad \text{where } n = 0, \pm 1, \pm 2, \pm 3, ...$$

because log(z) is a multifunction with an infinite number of values differing from one another by 2π. The fact that log is a multifunction means that any of the infinite number of values may be applied wherever log(z) is introduced. Hence, we introduce new variable names for the branch members chosen in each instance: m and k, where $k = m - n$.

$$\log(-z) = \ln r + i\,(\theta + 2n\pi + \pi)$$

$$\log(z) - \log(-z) = (\ln(r) + i\,(\theta + 2m\pi)) - (\ln(r) + i\,(\theta + 2n\pi + \pi))$$

$$= i\,(2m\pi - 2n\pi - \pi)$$

$$= 2k\pi - \pi \qquad \text{where } k = m-n$$

$$= \pi(2k-1)$$

This shows that the difference between log(z) and log(–z) is always an odd multiple of π. We conclude that log(–z) is always at a vertical distance of π from log(z). There can be no shared values.

What is wrong with this argument?

This argument can be hard to follow, so it will be broken down to correspond to equations, signified with E_1 to E_4, etc. and implications, which correspond to the lines, signified with (1), (2), (3).

$$E_1\text{: } \log\left[(-z)^2\right] = \log[z^2] \Longrightarrow E_2\text{: } \log(-z) + \log(-z) = \log(z) + \log(z) \quad (1)$$

$\Longrightarrow E_3$: 2 log(–z) = 2 log(z) (2)

$\Longrightarrow E_4$: log(–z) = log(z) (3)

Since log(z) is a multifunction with a many-to-one relationship to z, the equal sign must be given a different meaning from its normal one-to-one sense. The equal sign must be many-to-many. With log, both sides of the equation must contain exactly the same infinity of values. To write log(a) = log(b) is to assert that all possible values of log(a) also occur in log(b) and *vice versa*. In the following exercise the letters m, n, k, q, and p invoke the same infinity of values that were assigned to n in the first part of this exercise. The sums k = (m+n), p = (m + n + 1), s = (p+q), etc., cover the same infinity of integers as their parts do individually. Let $z = re^{i\theta}$, where r = |z|.

IMPLICATION (1)

To show that the implication in line (1) is wrong, we must show that $\log\left[(-z)^2\right] \neq$ log(-z) + log(-z) and/or $\log[z^2] \neq$ log(z) + log(z). *We expand* each side using log(z) = ln|z| + i arg(z). If we write this as (E_1 LHS = E_1 RHS) $\Longrightarrow$ (E_2 LHS = E_2 RHS), proving the implication will require showing that (E_1 LHS = E_2 LHS) and (E_1 RHS = E_2 RHS). Either side of an equation may have two log terms, each requiring a new integer variable for arg.

$$\ln r^2 + i\arg\left[(-z)^2\right] = \ln r^2 + i\arg[z^2]$$

E_1: $\mathbf{\ln r^2 + 2i\theta + m2\pi i = \ln r^2 + 2i\theta + n2\pi i}$ (because $(-z)^2 = z^2$)

Both sides include all possible multiples of 2πi and they are otherwise equal.

We can expand E_2 (using j, k, m, n, p, q)

$$\log(-z) + \log(-z) = \log(z) + \log(z)$$

$$[\ln r + i\arg(-z)] + [\ln r + i\arg(-z)] = [\ln r + i\arg(z)] + [\ln r + i\arg(z)]$$

$$[\ln r + i(\theta + \pi) + 2m\pi i] + [\ln r + i(\theta + \pi) + 2n\pi i] = [\ln r + i\theta + 2j\pi i] + [\ln r + i\theta + 2k\pi i]$$

$$2\ln r + 2i\theta + 2\pi i + (m+n)2\pi i = 2\ln r + 2i\theta + 4(j+k)\pi i$$

$$2\ln r + 2i\theta + (m+n+1)2\pi i = 2\ln r + 2i\theta + (j+k)2\pi i$$

E_2: $\mathbf{\ln r^2 + 2i\theta + p2\pi i = \ln r^2 + 2i\theta + q2\pi i}$, where p = (m+n+1), q = (j + k)

Now place E_1 and E_2 in sequence so they can be compared:

E_1: $\mathbf{\ln r^2 + 2i\theta + m2\pi i = \ln r^2 + 2i\theta + n2\pi i}$
E_2: $\mathbf{\ln r^2 + 2i\theta + p2\pi i = \ln r^2 + 2i\theta + q2\pi i}$

We can see that E_1 LHS = E_2 LHS and E_1 RHS = E_2 RHS, because the expressions on both sides have all possible multiples of 2πi.

IMPLICATION (2)

$E_2 \Longrightarrow E_3$, again writing the logs in expanded form. Since we already have the expanded form of E_2, we expand E_3:

E_3: $\mathbf{2\ (\ln r + i\theta + \pi i + m2\pi i) = 2\ (\ln r + i\theta + n2\pi i)}$ (expands 2 log(–z) = 2 log(z))

Now place E_2 and E_3 in sequence:

E_2: $\mathbf{\ln r^2 + 2i\theta + p2\pi i = \ln r^2 + 2i\theta + q2\pi i}$
E_3: $\mathbf{2\ (\ln r + i\theta + \pi i + m2\pi i) = 2\ (\ln r + i\theta + n2\pi i)}$

We wish to show that the LHS of line (1) E_2, which expands log(–z) + log(–z) equals the LHS of line (2) E_3, which expands 2 log(–z).

$2 \ln r + 2i\theta + p2\pi i\ ? = [2 \ln r + 2i\theta + 2i\pi + (2m)\ 2\pi i = 2 \ln r + 2i\theta + (2m+1)\ 2i\pi = 2 \log(-z)]$

We see that the $p2\pi i$ contains any number of multiples of $2\pi i$, but $(2m+1)\ 2\pi i$ contains only odd multiples of $2\pi i$. Then E_2 LHS $\neq$ E_3 LHS.

On the RHS, we compare E_2 RHS , which expands log(z) + log(z), to E_3 RHS, which expands 2 log(z), which can be written $2\ (\ln r + i\theta + 2n\pi i)$:

$2 \ln r + 2i\theta + q2\pi i\ ? = [2 \ln r + 2i\theta + (2n)2\pi i = 2 \log(z)]$

We see that the LHS has all possible multiples of $2\pi i$, while the RHS has only even multiples of $2\pi i$. Then E_2 RHS $\neq$ E_3 RHS. Combining this test with the previous test, the second implication (line 2) fails.