Needham, Chapter 2, Exercise 26, p. 118.
Gary Palmer, November 18, 2020, 4:50 pm PDST

*26. For each function f(z) below, find and then plot all the branch points and singularities. Assuming that these functions may be expressed as power series centred at k [in fact they can be], use the result (27) on p. 96 to verify the stated value of the radius of convergence R.*

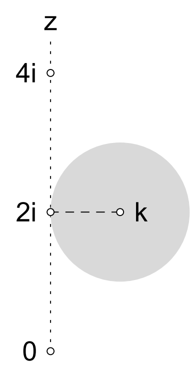



Figure 1. $f(z) = 1/(e^{\pi z}-1)$

*(i) If $f(z) = 1/(e^{\pi z} - 1)$ and $k = (1 + 2i)$, then $R = 1$.*

The singularities of f(z) occur where $e^{\pi z} = e^{n2\pi i} = 1$, which implies that $z = 2ni$, where n = 0, 1, 2, 3, .... Their number is infinite.

$$e^{\pi z} - 1 = e^{\pi 2ni} - 1 = 1 - 1 = 0$$

We want the nearest singularity to k = 1 + 2 i. The nearest singularity is at z = 2i. Then the radius of convergence R is $|2i - k| = |2i - (1+2i)| = 1$ as shown in Figure 1. The gray disc is the disc of convergence.

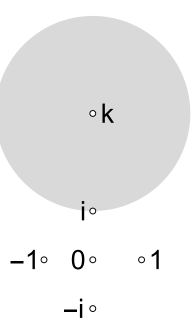



Figure 2. $f(z) = \sqrt[5]{z^4 - 1}$

*(ii) If f(z) is a branch of $\sqrt[5]{z^4 - 1}$ and $k = 3i$, then $R = 2$.*

$$f(z) = \sqrt[5]{z^4 - 1}$$

$$= \sqrt[5]{\left(z^2 - 1\right)\left(z^2 + 1\right)}$$

$$= \sqrt[5]{(z-1)(z+1)(z-i)(z+i)}$$

The branch points are 1, –1, i, –i. k = 3i is closest to i, so R = |3i – i| = |2i| = 2.

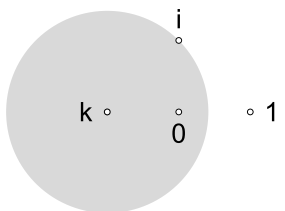
Out[•]=



Figure 3. f(z) = $\sqrt{z-i}$/(z–1)

*(iii) If f(z) is a branch of $\sqrt{z-i}$/(z–1) and k = –1, then R = $\sqrt{2}$.*

There is a branch point at z = i and a singularity at z = 1. k is closer to the branch point i than it is to the singularity 1, so there is a radius of convergence at |k–i| = |–1 – i| = $\sqrt{2}$ (Figure 3).