Needham, Chapter 2, Exercise 25, p. 118.
Gary Palmer, November 11, 2020, 6 pm

Out[•]=

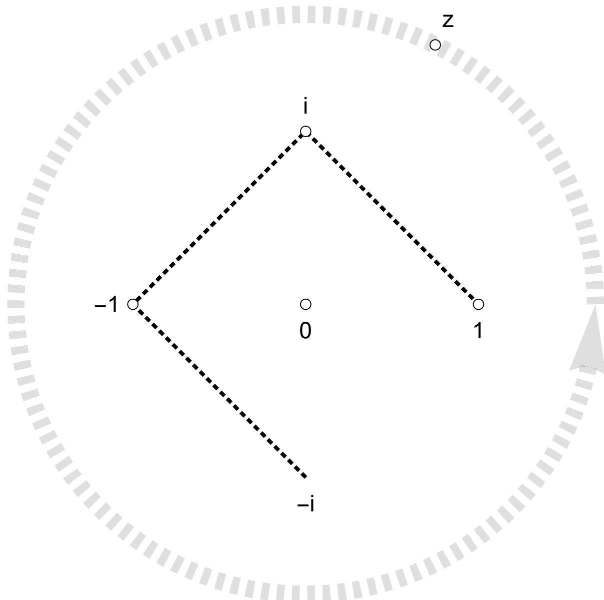

Figure 2. Traversal of z
around branch points 1,−1, i and −i

25. Describe the branch points of the function $f(z) = 1/\sqrt{1-z^4}$. What is the smallest number of branch cuts that may be used to obtain single-valued branches of f(z)? Sketch an example of such cuts. [...].

Rewrite f(z) as

$$f(z) = \frac{1}{\sqrt{(1-z)\,(1+z)\,(i-z)\,(i+z)}}$$

Then f(z) has four branch points {1, -1, i, -i}.

We rewrite f(z) using (z-p) = $r_\alpha\, e^{i\theta_\alpha}$ r = |z-p|, where α = 1, 2, 3, or 4, indexing arg(1-z), arg(1+z), ...

$$f(z) = \sqrt{r_1\, r_2\, r_3\, r_4}\, e^{i\left(\frac{\theta_1+\theta_2+\theta_3+\theta_4}{2}\right)}$$

$$= \mathrm{R}e^{i\left(\frac{\theta_1+\theta_2+\theta_3+\theta_4}{2}\right)}, \text{ where } \mathrm{R} = \sqrt{r_1\, r_2\, r_3\, r_4}$$

Let z traverse a loop round all four branch points:

$$f(ze^{i2\pi}) = \mathrm{R}e^{i\left(\frac{\theta_1+2\pi+2\pi\theta_2+\theta_3+2\pi+\theta_4+2\pi}{2}\right)} = \mathrm{R}e^{i\left(\frac{\theta_1+\theta_2+\theta_3+\theta_4}{2}\right)}\, e^{4\pi}$$

f(z) has returned to its starting point after a single traversal of z.

What is the smallest number of branch cuts that may be used to obtain single-valued branches of f(z)?

If the cut is continuous but not necessarily smooth, one cut suffices. The cut must pass through or touch all four points as in Figure 1, but the cut must not be a closed loop. Then the situation is analogous to that in [35b]. f(z), as any square root function, has two single-valued branches on this single finite branch cut.

Sketch an example of such cuts.

See Figure 1.