Needham, Chapter 2, Exercise 24, p. 117-118
G. Palmer, November 8, 2020, 3 pm PST

24 Consider the multifunction $f(z) = \sqrt{z-1}\ \sqrt[3]{z-i}$.

(i) Where are the branch points and what are their orders?
(ii) Why is it not possible to define branches using a single branch cut of the type shown in [35b]?
(iii) How many values does f(z) have at a typical point z? Find and then plot all the values of f(0).
(iv) Choose one of the values of f(0) which you have just plotted, and label it p. Sketch a loop L that starts and ends at the origin such that if f(0) is initially chosen to be **–***1, then as z travels along L and returns to the origin, f(z) travels along a path from –1 to p. Do the same for each of the other possible values of f(0).*

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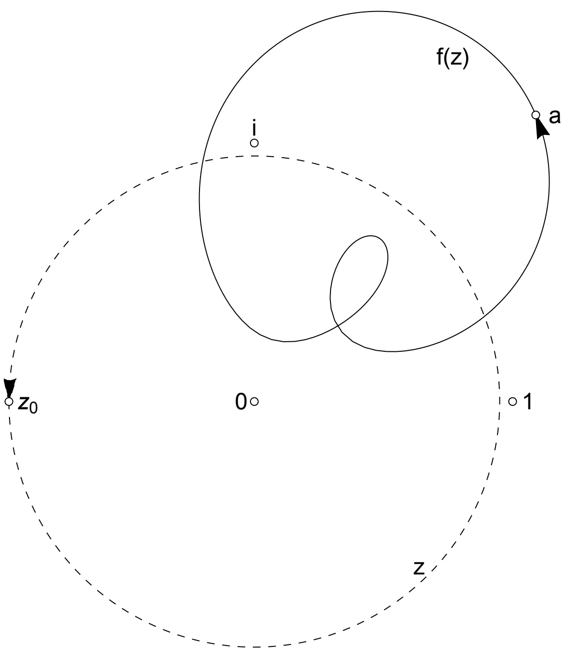

Figure 1. $f(z) = \sqrt{z-1}\ \sqrt[3]{z-i}$ on circle centered at origin with r = 0.95

(i) Where are the branch points and what are their orders?

On p. 92, we find that *a branch point z = q of f is a point such that f(z) fails to return to a as z travels along any loop that encircles q once*.

Needham does not give any criteria for identifying branch points of functions by inspection of the written function itself. We have to look at the results of the function. The discussion on pp. 95 centres on $\sqrt[3]{z}$, which has a branch point at 0. We note that 0 is also a zero-point for $\sqrt[3]{z}$. We could pick candidate branch points randomly and test them by letting z travel along a closed loop that encircled each candidate and observe whether f(z) returns to its starting point. Suppose that the origin is a candidate branch point for f(z) = $\sqrt{z-1}\ \sqrt[3]{z-i}$. Then letting z = $re^{i\theta}$, where r is any positive real number **less than 1**, suppose z makes a revolution to advance to $re^{i(\theta+2\pi)}$. This circle is a closed loop. At the same time, f(z) starts at $\sqrt{re^{i\theta}-1}\ \sqrt[3]{re^{i\theta}-i}$ and advances to $\sqrt{re^{i(\theta+2\pi)}-1}$ $\sqrt[3]{re^{i(\theta+2\pi)}-i}$. Both are located at the same point in the complex plane.

Figure 1 shows that f(z) returns to to its starting point **a** after one traversal of z around the circle starting at (–.95). Zero is not a branch point. One could plot five additional curves like that in Figure 1 based on combinations of the two roots of the radius of (z-1) and the three roots of the radius of (z-i). The pseudocode for the solid curve in Figure 1 is:

f(z) = {

arg = Arg(z-1)

θ_1 = **if** (0 < arg ≤ π) **then** arg

else (2π + arg)

θ_2 = Arg(z-i)

return $(z-1)^{1/2}(z-i)^{1/3}e^{i\theta_1/2}e^{i\theta_2/3}$

}

This is a somewhat laborious way to eliminate candidates for branch points. But at least we have discovered that the space around the origin extending out close to, but not including, r = 1 does not contain a branch point. It is easier to make the determination by drawing branch cuts and visualizing the rotations of z about points.

There is a way to determine the branch points by inspection of the written function. We know that $z^{1/n}$ applied to a loop encircling the origin does not return to the starting point for all n = 2, 3, 4, ..., because it's argument is a fraction of arg(z). Then it must have a branch point, which is the origin. Since z can be any complex number, this applies also to $\sqrt{z-1}$, $\sqrt[3]{z-i}$, and f(z) = $\sqrt{z-1}\ \sqrt[3]{z-i}$. Since z may be 0, the branch points must be 1 and i. In fact, to display the analogy of the square and cube roots of (z-1) or (z-i) with the nth root of z, we could have written $z^{1/n} = \sqrt[n]{z-0}$.

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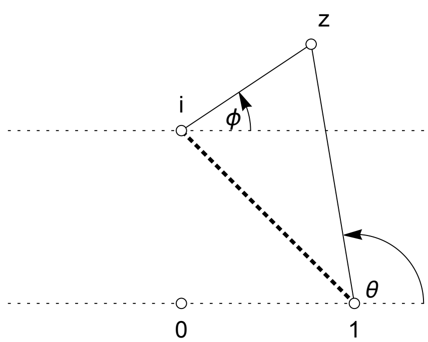

Figure 2. Rotation of z
around branch points 1 and i

There are six branches found as the product of the two square roots and the three cube roots. The order of branch point 1 is 1. The order of branch point i is 2.

I started by trying to reproduce Vasco’s answer, but I have reworked some of it according to my own notational preferences. Based on the function f(z) = $\sqrt{z-1}\ \sqrt[3]{z-i}$, rewrite

$$z - 1 = r_1\, e^{i\theta_1}, \qquad z - i = r_2\, e^{i\theta_2}$$

$$f(z) = \sqrt{r_1}\ \sqrt[3]{r_2}\ e^{i\,\frac{\theta_1}{2}} e^{i\,\frac{\theta_2}{3}}$$

θ_1 and θ_2 are the angles of z – 1 and z – i, not of z. We can define a branch cut extending from the branch point (1 or – i) to $-\infty$. These branch cuts define domains S_{θ_1} and S_{θ_2} where $-\pi < \{\ \theta_1,\ \theta_2\ \} \le \pi$. f(z) is the principal branch.

Let $f(z)$ be the value of the function on a closed loop L on the z-plane as shown in figure 2. "Let z travel anticlockwise round a closed loop that encloses only the branch point at z = 1." Then after one traversal of z about 1, beginning at z_0 and ending at z_L, we have

$$\begin{aligned} f(z_0) &= \left(\sqrt{r_1}\ \sqrt[3]{r_2}\ e^{i\,\frac{\theta_1}{2}} e^{i\,\frac{\theta_2}{3}}\right)[z_0] \\ f(z_L) &= \sqrt{r_1}\ \sqrt[3]{r_2}\ e^{i\,\frac{\theta_1+2\pi}{2}} e^{i\,\frac{\theta_2}{3}} \\ &= \sqrt{r_1}\ \sqrt[3]{r_2}\ e^{i\,\frac{\theta_1}{2}} e^{i\,\frac{\theta_2}{3}}\ e^{i\pi} \end{aligned}$$

$$= f(z_0)\, e^{i\pi}$$
$$= -f(z_0)$$

Since after one revolution of z, f does not return to its original location, 1 is a branch point of f.

If z traverses the loop again, then

$$f(f(z_L)) \;=\; -\,(-f(z_0)) = f(z_0)$$

so two complete traversals on a closed loop return f to its original value at z_0. This is what we expect for a square root function. z = 1 is a branch point of order 1, a simple branch point.

Repeat the exercise by sending z anticlockwise in a loop that encloses only i by adding 2π to θ_2.

$$f(z_L) \;=\; \sqrt{r_1}\;\sqrt[3]{r_2}\; e^{i\,\frac{\theta_1}{2}} e^{i\,\frac{\theta_2+2\,\pi}{3}}$$

$$= f(z_0)\, e^{i\,\frac{2\,\pi}{3}}$$

Three loops are required to return f to its original position, so i is a branch point of order 2. We illustrate the result of three loops by adding $6\,\pi$ to θ_2.

$$\sqrt{r_1}\;\sqrt[3]{r_2}\; e^{i\,\frac{\theta_1}{2}} e^{i\,\frac{\theta_2+6\,\mathrm{Pi}}{3}} = f(z_0)\, e^{i 2\pi}$$

This is a return to the same point in the plane as $f(z_0)$.

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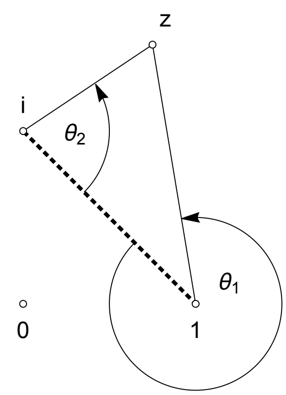


Figure 3. Domain when branch c
is segment (i–1)

(ii) Why is it not possible to define branches using a single branch cut of the type shown in [35b]?

Confining the cut line to the segment (i – 1) as shown with the bold dashed line in Figure 2, prevents the cuts from extending to infinity, but one can still extend the function to the whole plane.

As I read pp. 97 and 98, regarding the function $\sqrt{(z-i)(z+i)}$, it is possible to create two branches by letting z traverse a single closed loop that encircles both branch points. In that example, a traversal returns to the starting point. The two single-valued branches of the square root are created on the domain or set S defined by a single branch cut. Vasco wrote (p.c., Forum) that two branches are created by ensuring that z does not cross the cut.

But in this exercise where f(z) = $\sqrt{z-1}\ \sqrt[3]{z-i}$, if we draw a branch cut between i and 1, f does not return to the starting point after one revolution of z that encircles both points. f(z) returns to the starting point only after six loops of z. One can not define a single-valued branch based on a cut, because a branch cut by definition bounds a single traversal of z around a closed loop.

What would one call the boundaries of a cycle based on the number of traversals of z required to return a multifunction to its starting point?

(iii) How many values does f(z) have at a typical point z? Find and then plot all the values of f(0).

Since $\sqrt{z-1}$ is a two–valued multifunction and $\sqrt[3]{z-i}$ is a three–valued multifunction, f(z) has 2x3 = 6 values at a typical point z. A typical point is any point that is not a branch point or a singularity.

Zero is a typical point.

To plot the values of f(0), we note the radius of z as it traverses the unit circle as R = 1. Then we find the arguments when z = 0. The argument of (z–1) is θ_1 = Arg(–1) = π, and the argument of (z-i) is θ_2 = Arg($-i$) = $-\pi/2$. Then, the value of the principal branch at z = 0 is:

$$f_1(0) = \sqrt{r_1}\ \sqrt[3]{r_2}\ e^{i\theta_1/2}\ e^{i\theta_2/3} = e^{i(\theta_1/2+\theta_2/3)} = e^{i(\pi/2-\pi/6)} = e^{i\pi/2}\ e^{-i\pi/6} = e^{i\pi/3}$$

$$\{\{0,0\},\{2\pi,4\pi\},\{0,2\pi\},\{2\pi,0\},\{0,4\pi\},\{2\pi,4\pi\}\}$$

$e^{i\pi/3}$ is the first of the 6th roots of unity shown in Figure 4[a]. As we add loops to θ_1 and θ_2, we will highlight the changes in the list below the derivation. The content of these lists are included with other values in Table 1, below.

When we rotate z about 1 in [$\sqrt{z-1}$] to obtain a new branch, we add 2π to the angle of θ_1 in $f_0(0)$ and figure 1. We do this by multiplying by $e^{i2\pi/2} = e^{i\pi}$. The result in is the fourth root of unity.

$$f_4(0) = \left(e^{i\pi/2}\ e^{i\pi}\right) e^{-i\pi/6} = e^{-i2\pi/3} \qquad \text{(one loop of z about 1)}$$

$$\{\{0,0\},\{2\pi,4\pi\},\{0,2\pi\},\{2\pi,0\},\{0,4\pi\},\{2\pi,4\pi\}\}$$

A rotation of z about i as in $\sqrt[3]{z-i}$ adds 2π to θ_2, so we can add $2\pi/3$ and $4\pi/3$ to the angles of f_0 and f_1 to obtain more branches. We do this by multiplying by $e^{i2\pi/3}$ or $e^{i4\pi/3}$.

$$f_3(0) = f_2(0)e^{i2\pi/3} = e^{i\pi/2}\left(e^{-i\pi/6}e^{i2\pi/3}\right) = e^{i\pi} = -1 \qquad \text{(one loop of z about i)}$$

$$\{\{0,0\},\{2\pi,4\pi\},\{0,2\pi\},\{2\pi,0\},\{0,4\pi\},\{2\pi,2\pi\}\}$$

$$f_6(0) = f_3(0)e^{i\pi} = \left(e^{i\pi/2}\ e^{i\pi}\right)\left(e^{-i\pi/6}e^{i2\pi/3}\right) = e^{2\pi} = 1 \text{(one loop of z about 1 and one about } i\text{)}$$

$$\{\{0,0\},\{2\pi,4\pi\},\{0,2\pi\},\{2\pi,0\},\{0,4\pi\},\{2\pi,2\pi\}\}$$

$$f_5(0) = f_2(0)e^{i4\pi/3} = e^{i\pi/2}\left(e^{-i\pi/6}e^{i4\pi/3}\right) = e^{-i\pi/3} \qquad \text{(two loops of z about i)}$$

$$\{\{0,0\},\{2\pi,4\pi\},\{0,2\pi\},\{2\pi,0\},\{0,4\pi\},\{2\pi,4\pi\}\}$$

$$f_2(0) = f_5(0)e^{i\pi} = \left(e^{i\pi/2}\ e^{i\pi}\right)\left(e^{-i\pi/6}e^{i4\pi/3}\right) = e^{i2\pi/3} \qquad \text{(one loop of z about 1 and two about i)}$$

$$\{\{0,0\},\{2\pi,4\pi\},\{0,2\pi\},\{2\pi,0\},\{0,4\pi\},\{2\pi,4\pi\}\}$$

We arrange the roots into a list ordered by their position on the unit circle {$e^{i\pi/3}$, $e^{i2\pi/3}$, –1, $e^{-i2\pi/3}$,

$e^{-i\pi/3}$, 1} as in Figure 4[a]. The subscripts of f reflect their position in this list and on the circle displaying the roots of unity.

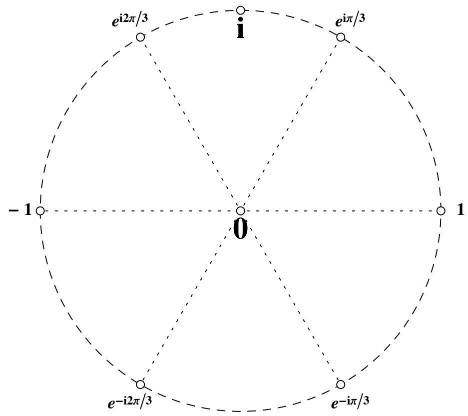
Vasco notes

If figure 3 looks familiar, this is because the 6 values of f(0) are the sixth roots of unity (See page 26 of the book, and chapter 1 exercise 34 on page 51). This can be shown as follows

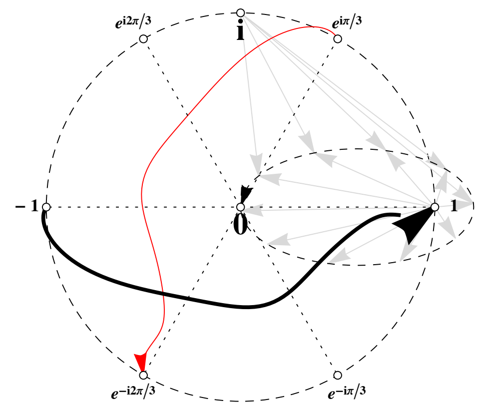
Define $a = f(0) = \sqrt{-1}\,\sqrt[3]{-i}$

So $a^6 = \left(\sqrt{-1}\,\sqrt[3]{-i}\right)^6 = (-1)^3\,(-i)^2 = (-1)(-1) = 1$

Vasco's Figure 3 has a formatting of the values that profiles the mirror imagery across the origin. I have found it helpful to profile angular distance from 1 in either direction.

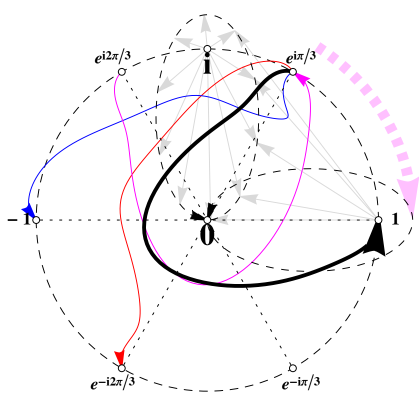


[a] The 6 values of f(0)

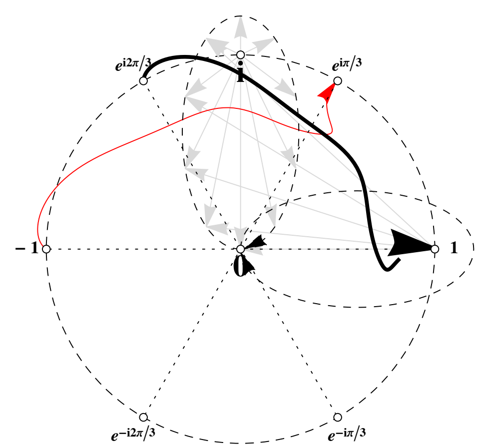


[b] Path from −1 to p = 1
$F_1(\mathrm{z})$ on ellipse about 1, anticlockwise
$F_1(\mathrm{z})\, e^{i2\pi/3}$

Out[•]=

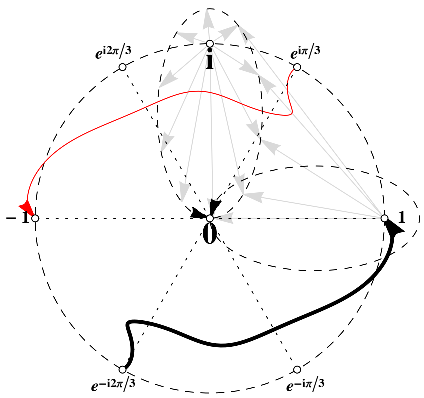


[c]: Path from $e^{i\pi/3}$ to p = 1;
$F_i(\mathrm{z})$ on antiClkWs loops
of ellipse about 1 and i
$F_1(\mathrm{z})\, F_i(\mathrm{z})\, e^{-i\pi/3}$

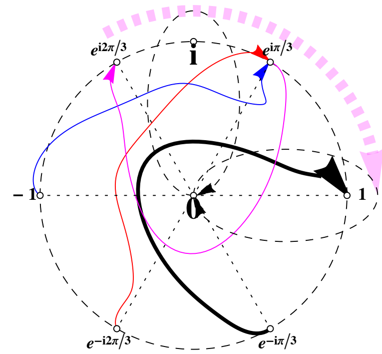


[d]: Path from $e^{i2\pi/3}$ to p = 1;
$F_i(\mathrm{z})$ on clockwise loop of ellipse about i
$F_i(\mathrm{z})\, e^{-i\pi/3}$

Figure 4. Pathways generated by traversal of z on closed loops [a] to [d]

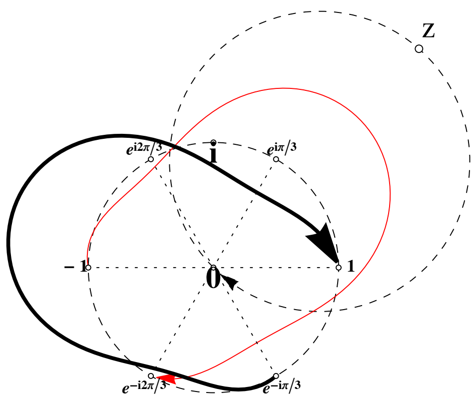


[e]: Path from $e^{-i2\pi/3}$ to p = 1;
F_i(z) on ellipse about i, anticlockwise
$-F_i$(z)



[f]: Path from $e^{-i\pi/3}$ to p = 1;
F_i(z) on ClkWs loop of ellipse about i
F_i(z) F_1(z) $e^{-i2\pi/3}$

Out[∘]=



[g]: Path from $e^{-i\pi/3}$ to p = 1;
F(z) on clockwise loop
round both branch points
F_i(z) $e^{i2\pi/3}$

Figure 4. Pathways generated by traversal of z on closed loops [e] to [g]

(iv) Choose one of the values of f(0) which you have just plotted, and label it p. Sketch a loop L that starts and ends at the origin such that if f(0) is initially chosen to be –1, then as z travels along L and returns to the origin, f(z) travels along a path from –1 to p. Do the same for each of the other possible values of f(0).

It appears to me that Needham has set up a very tricky problem. Exercise (iv) makes no mention of the fact that if f(0) = $\pi/3$ unless the calculation includes a rotation angle of $2\pi/3$ from $\pi/3$ to the chosen starting point (–1).

Looking at part (iii), we can see that the difference between $F_3(0) = -1$ and $F_6(z) = 1$ is one loop of z about the branch point at 1. That is, we must add 2π to θ_1 giving an increase of π. We should be able to find the path $F_1(z)$ leading to 1 by letting z traverse an ellipse that begins at 0 and loops around 1. In general, we can make a full rotation of 2π by letting z traverse 2 loops around 1 or 6 loops around i. We can change the direction of traversal by reversing the order of points on the loop, which can be accomplished in Mathematica by a statement such as Reverse @ loopPtsList. Since this also reverses the direction of the path, there may arise a need for rotations which restore the direction from starting point to target p.

$$f(\theta_1, \text{n}, \theta_2, \text{m}) = e^{i(\theta_1 + n\,2\,\pi)/2} e^{i(\theta_2 + m\,2\,\pi)/3}$$

Looking at part (iii), we can see that $f_3(0) = -1$ when $\theta_1 = \pi$ and $\theta_2 = (-\pi/2 + 2\pi)$. Following Vasco, we choose p = 1. As z travels anticlockwise from the origin, loops around branch point 1, and returns to the origin, f follows the path from –1 to 1. θ_1 makes a full revolution around branch point 1. θ_2 oscillates and returns to its original value with no net change as in Figure 4[b]. From Figure 2, when f(0) = –1, $\theta_1 = \pi$ and $\theta_2 = -\pi/2 + 2\pi = 3\pi/2$. See data line 2 of Table 1.

Table 1: Construction of sixth roots of unity by adding angles

θ_1	θ_2	$\theta_1/2$	$\theta_2/3$	$\theta_1/2 + \theta_2/3$	f(0)
π	$-\pi/2$	$\pi/2$	$-\pi/6$	$\pi/3$	$e^{i\pi/3}$
π	$-\pi/2 + 2\pi$	$\pi/2$	$\pi/2$	π	-1
π	$-\pi/2 + 4\pi$	$\pi/2$	$7\pi/6$	$-\pi/3$	$e^{-i\pi/3}$
$\pi+2\pi$	$-\pi/2$	$3\pi/2$	$-\pi/6$	$4\pi/3$	$-e^{i\pi/3}$ (or $e^{-i2\pi/3}$)
$\pi+2\pi$	$-\pi/2 + 2\pi$	$3\pi/2$	$\pi/2$	2π	1
$\pi+2\pi$	$-\pi/2 + 4\pi$	$3\pi/2$	$7\pi/6$	$2\pi/3$	$-e^{-i\pi/3}$ (or $e^{i2\pi/3}$)

We can wind z a full turn around either branch point. If we take a complete positive traversal around branch point 1, we increase θ_1 by 2π without changing θ_2 and *vice versa*: a traversal around branch point i augments θ_2 by 2π without changing θ_1. When we add 2π to θ_1, we increase Arg[f(0)] by only π. When we add $2\,\pi$ to θ_2 we increase Arg[f(0)] by only $2\pi/3$, so it takes three winds around branch point i to make a full revolution in f(0). The table describes f(0) on the rotation of z one or two full turns anticlockwise round branch points 1 or i in each of the multifunctions. The values of f(0) are the roots of unity.

That would be the ideal, but as the first paragraph of the answer to this exercise states, in order to compute a path one must also consider the computer's representation of the angles of (z-1) and (z-i) and the angle at f(0). In *Mathematica*, the function Arg[] returns angles from $-\pi$ to π. All the angles in the 3rd and 4th quadrant are given as negative numbers. We can write two functions, $F_1(z)$, which is designed for traversal of a closed loop round 1, and $F_i(z)$**,** which is for a closed loop around i or around both 1 and i. Both

F_1 and F_i utilize the formula $\sqrt{z-1}\ \sqrt[3]{z-i}\ e^{i(\theta_1/2+\theta_2/3)}$, where θ_1 = arg(z-1) and θ_2 = arg(z-i), but adjustments must be made to obtain continuous sequences of angles on a range of 2π. If we need to traverse loops around both 1 and i, we can find the inner product of the two path lists F_i(z) $F_1(z)$, as in figure 4[c]. The inner product essentially merges corresponding points in the lists representing F_α(z) on a loop. The following is a pseudocode representation of $F_1(z)$ and $F_i(z)$**,** which were used to plot the pathways shown in [b-e].

$F_1(z)$ = { # function for loop around 1

 r_1 = |z–1|; r_2 = |z–i|;

 θ_1 = 2π + arg(z – 1);

 θ_2 = arg(z – i);

 return $(z-1)^{1/2}\ (z-i)^{1/3}\ e^{i(\theta_1/2+\theta_2/3)}$

}

$F_i(z)$ = { # function for loop around i or around 1 and i

 r_1 = |z–1|; r_2 = |z–i|;

 θ_1 = arg(z – 1);

 θ_2 = **if** ($-\pi/2$ < arg(z – i) ≤ π),

  then arg(z – i)   # if z in quadrant IV, I, or II

  else 2π + arg(z – i) # if z in quadrant III, new branch

 return $(z-1)^{1/2}\ (z-i)^{1/3}\ e^{i(\theta_1/2+\theta_2/3)}$

}

In *Mathematica*, $F_1(z)$ and $F_i(z)$ can be written as modules. $F_1(z)$ traverses a loop round 1 and internally generates values θ_1 from π to 3π. $F_i(z)$ traverses a loop round i and internally generates principal values of θ_2 from $-\pi/2$ to $3\pi/2$.

Given the behavior of Mathematica Arg[], the default starting point is $f_1(0) = e^{i\pi/3}$. When z = 0, $\theta_1 = \pi$ and $\theta_2 = 3\pi/2$ (or $-\pi/2$ if confined to the principal values $-\pi < \theta \le \pi$).

Then $f_1(0) = e^{i\,(\theta_1/2 + \theta_2/3)} = e^{i\,(\pi/2 - \pi/6)} = e^{i\,\pi/3}$. This is the starting point for paths F(z) on closed loops, so if a different starting point is chosen arbitrarily, a rotation must be applied to F(z). For example, if we wish to start at f(0) = (-1), we have to apply an initial rotation of $2\pi/3$ by writing $F_1(z)\, e^{i 2\pi/3}$.

Paths such as Figure 4 [c], [f], and [g] from roots $\pi/3$ or $-\pi/3$ to branch point 1 cannot be plotted with a traversal of a single loop round either i or 1. Since they have $\pi/3$ radial separation from 1, one would hope to reach them by traversing a loop round i, but a full traversal about i only touches the unit circle at points that have $2\pi/3$ radial separation. Hence, one must traverse two loops: a loop round branch point 1 that moves $F_1(z)$ round zero by $\pm\pi$ and a loop round branch point i that rotates $F_i(z)$ round zero by an angle of $\pm 2\pi/3$ depending on the starting point and the target. If the starting point is $e^{-i 2\pi/3}$, then one should also apply an initial rotation of $\pm 2\pi/3$. Alternatively, one could do as Vasco has done and plot a loop encircling both branch points. See Figure 4[g].

Figure 4 [b-g] satisfies part (iv).

In 4[b], we first find and plot (red) $F_1(z)$, which begins at $\pi/3$. Then, by inspection, it appears that the path will fit the desired starting and ending points after a rotation of $2\pi/3$. Hence, the formula for the path is $F_1(z)\, e^{i 2\pi/3}$. Since $F_1(z)$ is a Mathematica list, this expression multiplies every point in the list by $e^{i 2\pi/3}$, effectively rotating the path by an angle of $2\pi/3$ about the origin.

In Figure 4[c], the branch points are traversed on two separate loops. The path of F_1 appears in red and the path of F_i appears in blue. Inspection of the paths show that both start at $e^{i\pi/3}$. The two paths, represented by two lists of complex numbers, are merged with the inner product F_1 (z) F_i (z), which begins at $e^{i 2\pi/3}$ and ends at $e^{i\pi/3}$. Since both paths begin at $e^{i\pi/3}$ and both have anticlockwise orientation, the merged path, shown in magenta, begins at $e^{i 2\pi/3}$, ends at $e^{i\pi/3}$, and has anticlockwise orientation. By inspection, one can see that a rotation of $e^{-i\pi/3}$, as shown by the thick dashed magenta arrow, brings the path into the alignment with the intended starting and ending points. Needham probably had in mind a path created with a single loop around the two branch points. For that, see 4[g].

Figure 4[d] The function F_i (z) has clockwise rotation (red), which means that it ends at $\pi/3$ instead of starting there. Then the desired path is obtained by rotating F_i (z) by $e^{-i\pi/3}$.

Figure 4[e], has anticlockwise traversal of the closed loop around i. $F_i(z)$ starts at $e^{i\pi/3}$ and proceeds to -1 by its curvy path shown in red. Then it is obvious by inspection that rotation of the path by π will plot the path with the desired endpoints.

Figure 4[f] involves a clockwise traversal about both branch points. As in 4[c], the path ends at $e^{i\pi/3}$ under clockwise traversal. The two lists of points from F_1 (z) in red and F_i (z) in blue are merged as the inner product F_1 (z) F_i (z). The merged path (magenta) under the inner product ends at $e^{i2\pi/3}$, so it must be rotated clockwise by that amount, as shown by the thick dashed magenta arrow.

Figure 4[g] involves a clockwise traversal about a closed loop (circle) that includes both 1 and i. The order of points on the circle was reversed to correspond to a clockwise traversal. Then one can see by inspection that rotation by $e^{i2\pi/3}$ brings the endpoints into alignment with the starting point and the target point of the path.

In[∘]:=