Needham, Chapter 2, Exercise 23, p. 117.
G. Palmer, October 3, 2020, 9:20 am PST

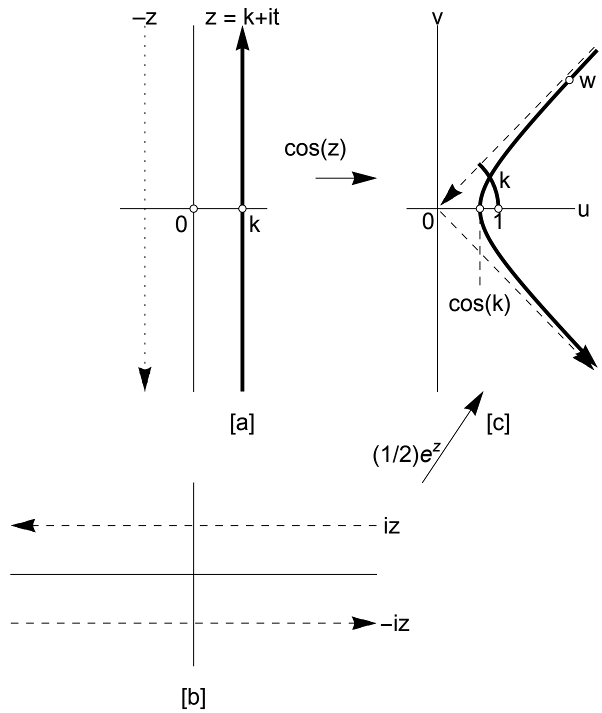

Figure 1. Hyperbola on x = k

23 If you did not do so earlier, sketch the image of a vertical line x = k under $z \mapsto w = \cos z$ by drawing the analogue of [26]. Deduce that the asymptotes of this hyperbola are arg w = ±k. Check this using the equation of the hyperbola.

To follow the analogy, our first image in the z-plane should show a vertical line with z traversing in the positive direction as in Figure 1[a]. We put this line at x = k. We could write this as z = (k + iy) or we could write z = (k + it), where k is constant and y or t varies. Using t rather than y profiles the continuous traversal in the positive direction along the bold line. Then we would plot a line in which z traverses in the opposite direction and –z = –(k + it). When we plot w = cos(z) on the bold line, the result is the hyperbola in [c].

The second step in the analogy is to rotate the first two lines in [a] by $\pi/2$ and plot them both under $(1/2)\,e^z$. The result is the two asymptotes shown in [c], which plot in the same order from top to bottom. As explained below, the value of z in $(1/2)\,e^z$ changes with the rotations and reversal of sines.

To follow the analogy to the final step, sum the asymptotes plotted with $(1/2)\,e^z$ to correspond to the sum of the two circles that produced the ellipse in [26]. We know the asymptotes extend in length from ∞ to 0 in the case of the UHP and from 0 to ∞ in the case of the LHP, but our plots are necessarily finite, so we let the parameter t vary from –2 to 2 to restrict the plotted points to finite length. Let the points w_U and w_L be the points traversing asymptotes on the Upper and Lower Half Planes, respectively. $w_U = (1/2)\, e^{\text{iz}}$ and $w_L = (1/2)\, e^{-\text{iz}}$. Under the change in t, w_U traverses the asymptote in the UHP by starting at $|w_U| = \infty$ or some arbitrary finite point and moving towards the origin. Under the same change in t, w_L starts at the origin and moves along the asymptote towards a point with infinite or finite length. At any moment in time, $w_U + w_L$ is a point on the hyperbola cos(z).

To understand why this works to reproduce the hyperbola, it helps to work out the algebraic result of the rotation and reversal of signs followed by the application of $(1/2)\, e^z$. After the reversal of sign and the rotation, we have two line segments that we may describe as iz and -iz. We apply $(1/2)\, e^z$ to both of these and sum the result:

$$(1/2)\, e^z[\text{iz}] = (1/2)\, e^{\text{iz}} \qquad (1)$$

$$(1/2)\, e^z[-\text{iz}] = (1/2)\, e^{-\text{iz}} \qquad (2)$$

$$(1) + (2) = \frac{e^{\text{iz}} + e^{-\text{iz}}}{2} = \cos(z)$$

Had we applied $(1/2)\, e^z$ directly to the unrotated lines, we would have had:

$$(1/2)\, e^z[z] + (1/2)\, e^z[-z] = \frac{e^z + e^{-z}}{2} = \cosh(z)$$

which would produce a different plot.

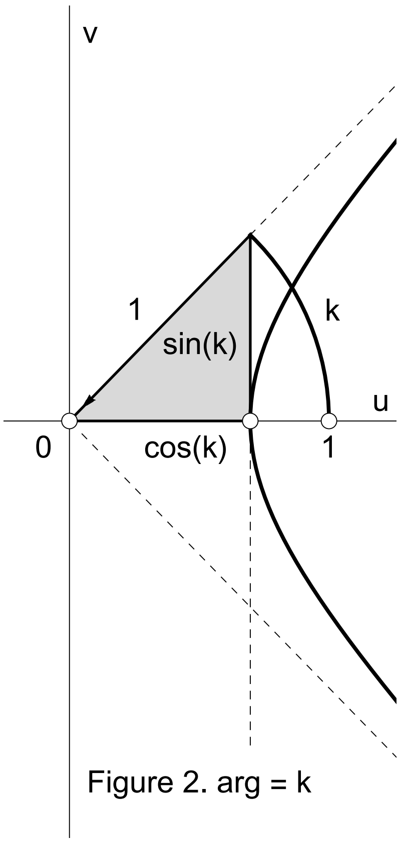

Figure 2. arg = k

Deduce that the asymptotes of this hyperbola are arg w = ±k.

One can deduce the arguments of the asymptotes from Figure 2. We know that $\cos^2(k) + \sin^2(k) = 1$ and the focus is 1, so the bold arc intersects the asymptote at a distance of 1 from the origin. Since we know the base to be cos(k), the height of the shaded triangle must be sin(k) and the angle of the triangle at the origin must be k. The asymptote in the LHP must have arg –k.

There is another way to deduce that the arguments of the asymptotes are ±k. Plot the asymptote in the UHP using (1).

$$\arg[(1/2)\, e^{i(k+\text{it})}] = \arg[(1/2)\, e^{-t}\, e^{\text{ik}}] = \text{k}$$

Plot the asymptote in the LHP with (2).

$$\arg[(1/2)\, e^{-i(k+\text{it})}] = \arg[(1/2)\, e^{t}\, e^{-\text{ik}}] = -\,\text{k}$$

Check this using the equation of the hyperbola.

This check is illustrated in Figure 2, which is a blowup from Figure 1[c]. Needham provided an equation for the hyperbola (p. 89).

$$(\text{u}/\cos x)^2 - (\text{v}/\sin x)^2 = 1$$

We can rewrite this for x at k.

$$(\mathrm{u}/\cos k)^2 - (\mathrm{v}/\sin k)^2 = 1$$

After solving for v and letting u = $\lim_{u\longrightarrow\infty} \sqrt{u^2 - \cos^2 k}$, we can write

$$\mathrm{v} = \pm \frac{\sin k}{\cos k}\, \mathrm{u} = \pm \tan(\mathrm{k})\, \mathrm{u}$$

$$\arg(\mathrm{v}/\mathrm{u}) = \tan^{-1}(\pm \tan \mathrm{k}) = \pm \mathrm{k}$$