Needham, Chapter 2, Exercises 22, pp. 117
G. Palmer, September 28, 2020, 2:10 pm PST

22. Reconsider the formula,

$$e^z = lim_{n\to\infty} P_n(z), \quad where \quad P_n(z) = (1 + z / n)^n.$$

(i) Check that $P_n(z)$ is the composition of a translation by n, followed by a contraction by (1/n), followed by the power mapping $z \mapsto z^n$.

Translate z by n:

$$T_n(z) \ = \ z \ + \ n$$

Contract $T_n(z)$ by 1/n. Let $C_{1/n}$(z) = z/n.

$$C_{1/n} \circ T_n(z) = \frac{z+n}{n} = 1 + \frac{z}{n}$$

Map $C_{1/n} \circ T_n(z)$ to it's own nth power:

$$P_n(z) \ = \ (C_{1/n} \circ T_n(z))^n = \left(1 + \frac{z}{n}\right)^n$$

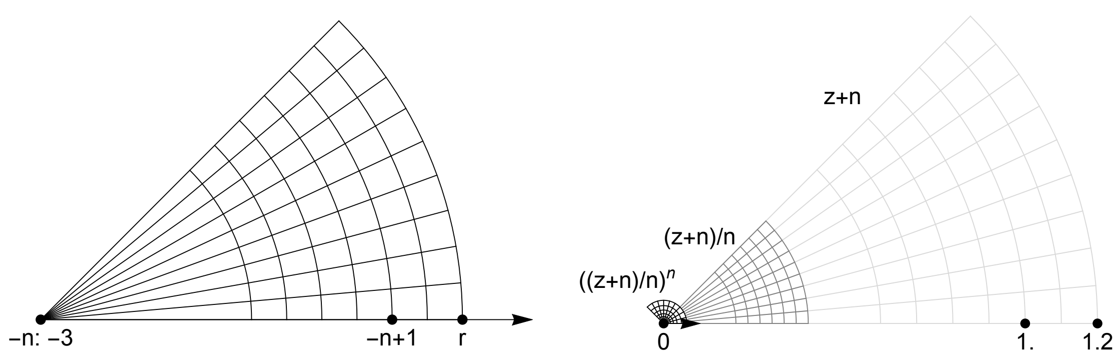

Figure 1. Modification of [4], p. 58

(ii) Referring to figure [4] on p. 58, use the previous part to sketch the images under $P_n(z)$ of circular arcs centred at – n, and of rays emanating from – n.

We do our sketching with a *Mathematica* program. In Figure 1a, the distance between arcs is (.1). The discussion on pp. 57-58 indicates that n is a positive integer. Draw circles and rays centred at – n on the z plane. Apply $P_n(z)$ to obtain their images on the w plane where w = $P_n(z) = \left(1 + \frac{z}{n}\right)^n$. C_{T_n} returns it to the neighborhood near the origin. We choose a small number, 3 for the translation. With a larger number, like 100, applying the power in the third stage shrinks this image to almost nothing. We apply the three stages of part (i) in sequence. Each stage is labeled in Figure 1. Stage 1, z ⟶ (z+n), is plotted in light gray, stage 2, z ⟶ z/n, is plotted in gray, and stage 3, z ⟶ z^n is plotted in black. Stages 2 and 3 shrink the rays and the radii of the circles as expected.

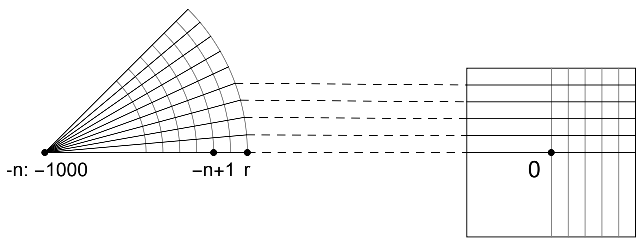

Figure 2. Projection to unit square

(iii) Let S be an origin-centred square (say of unit side) in the z-plane. With a large value of n, sketch [map] *just those portions of the arcs and rays (considered in the previous part) that lie within S.*

Given just the suggested model of [4] on p. 58, which gives us a pie slice with a grid like that of Figure 2a, one might question whether the arcs and rays will actually intersect S, which in this figure is n units from the centre of the pie slice to the center of S. With an n of 10^3, that would be 10^3 units distance, so the sides of the squares in the pie grid would be multiplied accordingly. Following Vasco, one can imagine that the rays extend all the way to the unit square and beyond, crossing it on the way. Then, if the angles of the rays are made sufficiently small, one would see approximations of horizontal lines crossing the square. Similarly, if more arcs are added at radial distances in the neighborhood of –.0001, one would see approximations of vertical lines crossing the square. The end result is a grid within the square. The sides of the squares in the grid of Figure 2b are approximately .1 unit so the invisible squares that they were projected from must have sides of .0001 unit.

In view of the following part (iv), we would like to see arcs with r < 1 projected to the left side of the square and arcs with radius > 1 projected to the right side of the square. There are no arcs with 1– .0005 ≤ r < 1 and 1 < r ≤ 1.0005 plotted in Figure 2a, but we can conjure them up. In particular, using appropriate expansions, we arrange for five arcs centred at (–n) from r = (1 – .0005) to r = (1 – .0001) project to five vertical lines on the LHS of the square. Five arcs from r = (1 + .0001) to r = (1 + .0005) project to five vertical lines on the RHS. The distances between these radii are so tiny, that were we to include these imagined arcs in the plot of Figure 2a, they would probably look like a slightly thicker arc intersecting the real axis at – n +1. In exercise (v), we find that the function f(z) = $(1 + z/n)^n$ reverses this projection, sending squares on a grid to approximate squares with sides formed by arcs and rays.

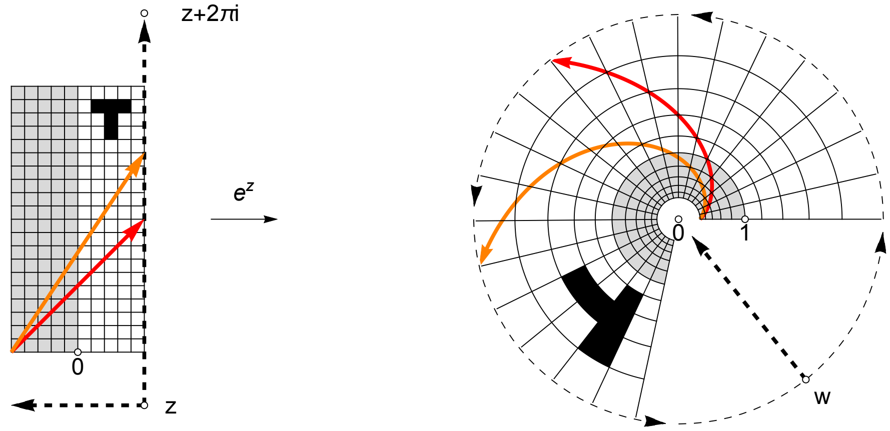

Figure 3. Facsimile of [19], p. 81

(iv) Use the previous two parts to qualitatively explain figure [19] on page 81.

Figure 3 bears important similarities to Figure 2, except that the order of figures a and b is reversed and they are placed in the two planes z and w, in contrast to Figure 2, which used only the z plane. The projection from the pie-grid to the grid square in Figure 2 is replaced by the function f(z) = $(1 + 1/z)^n$ in Figure 3. Instead of constructing an imaginary series of arcs in the neighborhood of the y-axis, as was done in part (iii), the function calculates the precise radii of arcs and angles of rays. The radii of the arcs are determined in part by each of the subfunctions in the analysis of f(z): z ↦ z+n, z ↦ z/n and z ↦ z^n. The angles of the rays are strongly influenced by the third subfunction: z ↦ z^n, which multiplies arg(z) by n.

Let z traverse any of the straight line segments in Figure 3a, including the sides of squares and the slanted arrows. Let the direction of traversal follow the direction of the dashed and colored arrows. The orientation of the black T reveals that the columns of Figure3a map to the white and shaded disc shape sectors of Figure 3b. The vertical lines map to circles and the horizontal lines map to rays. The left edge of the rectangle in Figure 3a maps to the inside circle bounding the shaded disc in Figure 3b. The y-axis of Figure 3a maps to the outside circle bounding the shaded disc in Figure 3b. The right edge of the rectangle in Figure 3a maps to the outside circle of Figure 3b. The series of arrows labeled z at the bottom of Figure 3a map to the arrows labeled w in Figure 3b. The real axis of Figure 3a maps to a segment of the positive real axis in Figure 3b. Red maps to red and orange maps to orange. Points west of the y-axis map to the inside of the unit disc. Points east of the y-axis map to the outside of the unit disc.

In the following, we recognize that f(z) = $(1 + z/n)^n$ converges to e^z, with the result that squares map to approximate squares and lines map to rays or circles. Angles between lines are preserved because the translation (z+n) and the contraction (z/n) are conformal functions (p. 130). z^n is also conformal because it is a rotation and an amplification. See chapter 4 for more on this.

Using the three steps of the previous parts to explain the figure, we see that z ⟶ n + z, where n in this case is 10^5, but it could also be ∞. We can multiply this by the length of a side. This means that n is way out of the field of view. Unlike in part (i), the figure in Figure 2a is centred horizontally at 0, not –n. Then the next step (n+z) ⟶ (n+z)/n would bring z back into a reasonable field of view. arg((n+z)/n)) = arg(1 + z/n). At this point, z has been transformed to 1 plus a very small complex number (z/n) with the very small angle ~y/1. Then (1+z/n) ⟶ $(1+z/n)^n$, which means the argument of (1+z/n) is multiplied by n, so the argument becomes approximately ny. This is what accounts for the greater rotation with distance from the real line.

Consider the red and orange arrows in Figure 3a that intersect a line parallel to the y-axis. The higher the intersection with the vertical line (higher y value), the larger its angle with the x-axis and the more the rays are rotated by $(1+z/n)^n$. The result is that the orange arrow appears to bend farther away from the red arrow. Vertical lines bend around the origin and they bend farther at the top (where the angle with the x-axis is greater). When the formula is applied to rays, the result is concentric circles. The same effect makes the rays farther apart on the outer rim than the inner rim. This accounts for the fact that the gray and white columns in Figure 2a map to segments of gray and white discs in Figure 3b.

How does $P_n(z)$ explain the transformation of horizontal lines to rays.

Moving along a horizontal line has a different effect depending on the distance from the x-axis. Points lying on the x-axis have arg = 0, so $(1+z/n)^n$ has no rotation there. Points in horizontal lines far from the x axis have relatively large angles which are multiplied by n, so they map to rays at regular intervals around the origin. The final angle of each ray formed under a given power n is determined by the y-value of its preimage.

How does $P_n(z)$ explain the transformation of slanted lines to spirals?

Moving along a slanted line is equivalent to varying x and y (or r and θ, where z = r$e^{i\theta}$) at the same time. The slanted lines in the z plane have positive slope. Hence, the result in the w plane is neither a ray (decreasing e^x and holding y constant in the preimage) nor a circle (increasing y and holding x constant in the preimage). It is instead, as y and x both increase, a curve anchored at $P_n(-5)$ with increasing positive angle and distance from the origin.