Needham, Chapter 2, Exercise 21, pp. 117
G. Palmer, September 18, 2020, 12:20 pm PST

21 Do the following problems by first substituting $z = r\, e^{i\theta}$ into the power series for e^z, then equating real and imaginary parts.

From p. 79,

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{(re^{i\theta})^n}{n!} = \sum_{n=0}^{\infty} \frac{r^n\, e^{in\theta}}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{r^n}{n!}(\cos(n\theta) + i\sin(n\theta)) \qquad \text{(note that } 0! = 1\text{)}$$

The real and imaginary parts of this series are

$$\mathrm{Re}\left(e^{re^{i\theta}}\right) = \mathrm{Re}\left(e^{r(\cos\theta + i\sin\theta)}\right) = \mathrm{Re}\left(e^{r\cos\theta}\, e^{i\,r\sin\theta}\right)$$

$$= \mathrm{Re}[\, e^{r\cos\theta}(\cos(r\sin\theta) + i\sin(r\sin\theta)\,]$$

$$= e^{r\cos\theta}\cos(r\sin\theta) = \sum_{n=0}^{\infty} \frac{r^n\cos n\theta}{n!} \qquad (1)$$

$$\mathrm{Im}\left(e^{re^{i\theta}}\right) = e^{r\cos\theta}\sin(r\sin\theta) = \sum_{n=0}^{\infty} \frac{r^n\sin n\theta}{n!} \qquad (2)$$

(i) Show that the Fourier series for $[\cos(\sin\theta)]\, e^{\cos\theta}$ is $\sum_{n=0}^{\infty} \frac{\cos n\theta}{n!}$, and write down the Fourier series for $[\sin(\sin\theta)]\, e^{\cos\theta}$.

The easy way to answer this is to let r = 1 in (1) and (2), and the answer falls out immediately. From (2)

$$[\sin(\sin\theta)]\, e^{\cos\theta} = \sum_{n=0}^{\infty} \frac{\sin n\theta}{n!}$$

(ii) Deduce that $\int_0^{2\,Pi} e^{\cos\theta}[\cos(\sin\theta)]\cos m\theta\, d\theta = (\pi/m!)$, where m is an integer.

$$\cos(\sin\theta)\, e^{\cos\theta} = \sum_{n=0}^{\infty} \frac{\cos n\theta}{n!}, \quad \text{by (i)}$$

$$\int_0^{2\,\mathrm{Pi}} e^{\cos\theta}[\cos(\sin\theta)]\cos m\,\theta\, d\theta = \int_0^{2\,\mathrm{Pi}} \sum_{n=0}^{\infty} \frac{\cos n\theta}{n!}\cos m\theta\, d\theta$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\int_0^{2\,\mathrm{Pi}} \cos n\theta\cos m\,\theta\, d\theta = \frac{\pi}{m!} \qquad \text{(by Exercise 20)} \quad \text{QED}$$

But if that is the case

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^{\cos\theta}[\cos(\sin\theta)] \cos m\theta \, d\theta = \frac{1}{\pi}\frac{\pi}{m!} = \frac{1}{m!}$$

(iii) By writing $x = (r/\sqrt{2}\,)$, find the power series for $f(x) = e^x \sin x$.

My first answer attempted to use exponentials and then multiply the sine series by $e^{r/\sqrt{2}}$ to obtain a new sine series. *I used Mathematica*, which gave a nice series in r, but which I suspect simply used the series for $e^{r/\sqrt{2}}$ in the product, which anticipates part (iv).

$$\sin x = \frac{e^{ir/\sqrt{2}} - e^{-ir/\sqrt{2}}}{2i}$$

$$e^x \sin x = e^{r/\sqrt{2}} \sin\left(\frac{r}{\sqrt{2}}\right) = e^{r/\sqrt{2}} \frac{e^{ir/\sqrt{2}} - e^{-ir/\sqrt{2}}}{2i}$$

Vasco has a better solution. We already have a solution for $e^{r\cos\theta}\sin(r \sin\theta)$, and we know from our definition of z that $x = r\cos\theta$. Then, since we are given that $x = r/\sqrt{2}$, it follows that $\cos\theta = 1/\sqrt{2}$, so $\theta = \pi/4 = \sin\theta$, and $r = \sqrt{2}\,x$. Substitute these values into equation (2):

$$e^{r\cos\theta}\sin(r \sin\theta) = \sum_{n=0}^{\infty} \frac{r^n \sin n\theta}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{\left(\sqrt{2}\,x\right)^n \sin[n\,(\pi/4)]}{n!} = \sum_{n=0}^{\infty} \frac{\left(\sqrt{2}\right)^n \sin[n\,(\pi/4)]}{n!} x^n$$

The first 6 terms of this series will be used for comparison in part (iv).

$$0 + \frac{\left(\sqrt{2}\right)^1 \sin[\pi/4]}{1!} x + \frac{2\sin[\pi/2]}{2!} x^2 + \frac{2\sqrt{2}\sin[3\pi/4]}{3!} x^3 + \frac{4\sin[\pi]}{4!} x^4 + \frac{4\sqrt{2}\sin[5\pi/4]}{5!} x^5 \cdots$$

$$= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \frac{x^6}{90} - \frac{x^7}{630} \cdots \qquad (3)$$

(iv) Check the first few terms of the series for f(x) by multiplying the series for e^x and $\sin x$.

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$e^x \sin x = (1 + x + \frac{x^2}{2!} + \frac{1}{3!}x^3 + \ldots)\,(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots)$$

$$= x \qquad - \frac{x^3}{3!} \qquad\qquad + \frac{x^5}{5!} \qquad \cdots$$

$$+ x^2 \quad + \frac{x^3}{2!} \quad - \frac{x^4}{3!} \qquad\qquad \cdots$$

$$+ \frac{x^4}{3!} \quad - \frac{x^5}{2!\;3!} \qquad \cdots$$

$$+ \frac{x^5}{4!} \qquad \cdots$$

$$= x + x^2 + \left(-\frac{x^3}{3!} + \frac{x^3}{2!}\right) + \left(-\frac{x^4}{3!} + \frac{x^4}{3!}\right) + \left(\frac{x^5}{5!} - \frac{x^5}{2!\;3!} + \frac{x^5}{4!}\right) + \cdots$$

$$= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \cdots$$

This series matches series (3) .

(v) Calculate the n^{nth} derivative $f^{(n)}(0)$ using (14) on p. 22 of Chapter 1. By using these derivatives in Taylor's Theorem, verify your answer to part (iii).

$$\frac{d^n}{dt^n}[e^{at} \sin bt] = \left(a^2 + b^2\right)^{\frac{n}{2}} e^{at} \sin\left[bt + n \tan^{-1}(b/a)\right] \qquad (14)$$

$$f(x) = e^x \sin x$$

If we let a = b = 1, then if we let t = x, the equation becomes

$$\frac{d^n}{dx^n}[e^x \sin x] = \left(\sqrt{2}\right)^n e^x \sin[x + n\,\pi/4]$$

Since we are to calculate the derivatives at x = 0, we can write

$$\frac{d^n}{dx^n}[e^x \sin x] = \left(\sqrt{2}\right)^n \sin[n\,\pi/4]$$

$$f^{(0)}(0) = \sin[0] = 0$$

$$f^{(1)}(0) = \sqrt{2}\, \sin[\pi/4] = 1$$

$$f^{(2)}(0) = 2 \sin[\pi/2] = 2$$

$$f^{(3)}(0) = 2\sqrt{2}\, \sin[3\,\pi/4] = 2$$

$f^{(4)}(0) = 4 \sin[\pi] = 0$

$f^{(5)}(0) = 4\sqrt{2} \sin[5\pi/4] = -4$

$f^{(6)}(0) = 8 \sin[3\pi/2] = -8$

$f^{(7)}(0) = 8\sqrt{2} \sin[7\pi/4] = -8$

Taylor expansion of F(x) at a of $e^x \sin x$.

$$\mathrm{F(a)} = \mathrm{f(a)} + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \cdots$$

$$\mathrm{F(0)} = \mathrm{f(0)} + \mathrm{f'(0)x} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots \qquad \text{(where a = 0)}$$

$$= 0 + \mathrm{x} + \frac{2x^2}{2!} + \frac{2x^3}{3!} + \frac{0x^4}{4!} + \frac{-4x^5}{5!} + \frac{-8x^6}{6!} + \frac{-8x^7}{7!} \cdots$$

$$= 0 + x + x^2 + \frac{x^3}{3} + 0 - \frac{x^5}{30} - \frac{x^6}{90} - \frac{x^7}{630} + 0 \cdots$$

This verifies the answer to part (iii). So we can write

$$e^x \sin \mathrm{x} = \sum_{n=0}^{\infty} \frac{\left(\sqrt{2}\right)^n \sin[n\,\pi/4]\, x^n}{n!}$$