Needham, Chapter 2, Exercise 20, pp. 116.
G. Palmer, September 13, 2020, 12:20 pm PST

20 Show that if m and n are integers, then $\int_0^{2\pi} \cos m\theta \cos n\theta \, d\theta$ *vanishes unless m = n, in which case it equals π. Likewise, establish a similar result for* $\int_0^{2\pi} \sin m\theta \sin n\theta \, d\theta$*. Use these facts to verify (19), at least formally.*

A calculation using *Mathematica* supports the premise of the exercise.

$m \neq n, \int_0^{2 \mathrm{Pi}} \mathrm{Cos}[m\,\theta]\,\mathrm{Cos}[n\,\theta] d\theta\text{:}\ 0$

$m = n, \int_0^{2 \mathrm{Pi}} \mathrm{Cos}[n\,\theta]\,\mathrm{Cos}[n\,\theta] d\theta\text{:}\ \pi$

$m \neq n, \int_0^{2 \mathrm{Pi}} \mathrm{Sin}[m\,\theta]\,\mathrm{Sin}[n\,\theta] d\theta\text{:}\ 0$

$m = n, \int_0^{2 \mathrm{Pi}} \mathrm{Sin}[n\,\theta]\,\mathrm{Sin}[n\,\theta] d\theta\text{:}\ \pi$

Rewrite cos(mθ) cos(nθ) as an exponential function, find the exponential product, and rearrange to obtain a result in terms of cosines, which can be easily integrated.

$$\cos(m)\cos(n\theta) = \frac{e^{im\theta}+e^{-im\theta}}{2} \cdot \frac{e^{in\theta}+e^{-in\theta}}{2} = \frac{e^{i(m+n)\theta}+e^{i(m-n)\theta}+e^{i(n-m)\theta}+e^{-i(m+n)\theta}}{4}$$

$$= \frac{1}{2}\left(\frac{e^{i(m+n)\theta}+e^{-i(m+n)\theta}}{2} + \frac{e^{i(m-n)\theta}+e^{-i(m-n)\theta}}{2}\right) = \frac{1}{2}(\cos(m+n)\,\theta + \cos(m-n)\,\theta)$$

Integrate the result.

$$\frac{1}{2}\int_0^{2\pi}[\cos(m+n)\,\theta + \cos(m-n)\,\theta]\,d\theta = -\frac{1}{2}\left[\frac{\sin(m+n)\,\theta}{m+n} + \frac{\sin(m-n)\,\theta}{m-n}\right]_0^{2\pi} = 0$$

because the sine of any integer multiple of π or 0 equals 0. Hence, *if m and n are integers, then* $\int_0^{2\pi} \cos m\theta \cos n\theta \, d\theta$ *vanishes.*

If m = n, the integral becomes

$$\frac{1}{2}\int_0^{2\pi}[\cos(2\,n)\,\theta + \cos(0)\,\theta]\,d\theta = \frac{1}{2}\int_0^{2\pi}[\cos(2\,n)\,\theta + 1]\,d\theta$$

$$\frac{1}{2}\left[\frac{\sin(2\,n)\,\theta}{2\,n} + \theta\right]_0^{2\pi} = \frac{1}{2}\,(0 + 2\pi) = \pi$$

Hence, $\int_0^{2 \mathrm{Pi}} \cos m\theta \cos n\theta \, d\theta = \pi$ when m = n. This is what we wanted.

We repeat the exercise for $\int_0^{2\pi} \sin m\theta \sin n\theta \, d\theta$.

For m ≠ n

$$\sin(m\theta)\,\sin(n\theta) = \frac{e^{im\theta}-e^{-im\theta}}{2i}\cdot\frac{e^{in\theta}-e^{-in\theta}}{2i} = \frac{e^{i(m+n)\theta}-e^{i(m-n)\theta}-e^{i(n-m)\theta}+e^{-i(m+n)\theta}}{4(-1)}$$

$$= -\frac{1}{2}\left(\frac{e^{i(m+n)\theta}+e^{-i(m+n)\theta}}{2} - \frac{e^{i(m-n)\theta}+e^{-i(m-n)\theta}}{2}\right) = -\frac{1}{2}(\cos(m+n)\,\theta - \cos(m-n)\,\theta)$$

Integrate the result.

$$-\frac{1}{2}\int_0^{2\pi}[\cos(m+n)\,\theta - \cos(m-n)\,\theta]\,d\theta = -\frac{1}{2}\left[\frac{-\sin(m+n)\,\theta}{m+n} + \frac{\sin(m-n)\,\theta}{m-n}\right]_0^{2\pi} = 0$$

Where n = m, the integral becomes

$$\frac{1}{2}\int_0^{2\pi}[\cos(2\,n)\,\theta + \cos(0)\,\theta]\,d\theta = \frac{1}{2}\int_0^{2\pi}[\cos(2\,n)\,\theta + 1]\,d\theta$$

$$\frac{1}{2}\left[\frac{\sin(2\,n)\,\theta}{2\,n} + \theta\right]_0^{2\pi} = \frac{1}{2}\,(0 + 2\pi) = \pi$$

Hence, $\int_0^{2\,\mathrm{Pi}} \sin m\theta \sin n\theta \, d\theta \;=\; \pi$ when m = n. This is a similar result. It is, in fact, the same result.

Use these facts to verify (19), at least formally.

According to Vasco (p.c. Forum), the variable "n" in a_n and b_n is a particular positive integer. It is not necessarily the same as the n in the RHS of F(θ), so we take it to be a constant. Accordingly, F(θ) is rewritten here as $\mathcal{F}(\theta)$, and (19) is rewritten as (19)*.

$$\mathcal{F}(\theta) = \frac{1}{2}\,a_0 + \sum_{m=1}^{\infty}[a_m \cos m\theta + b_m \sin m\theta]$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi}\mathcal{F}(\theta)\cos n\theta\,d\theta, \quad b_n = \frac{1}{\pi}\int_0^{2\pi}\mathcal{F}(\theta)\sin n\theta\,d\theta \qquad (19)^*$$

Substitute for $\mathcal{F}(\theta)$ in the equation for a_n and multiply through by cos nθ.

$$\frac{1}{\pi}\int_0^{2\pi}\mathcal{F}(\theta)\cos n\theta\,d\theta = \frac{1}{\pi}\int_0^{2\pi}\Big[\frac{1}{2}\,a_0 \cos n\theta\,d\theta + \sum_{m=1}^{\infty}[a_m \cos m\theta \cos n\theta + b_m \sin m\theta \cos n\theta]\,d\theta$$

Rearrange the sum, integrals, and coefficients.

$$= \frac{1}{\pi}\Big[\int_0^{2\pi}\frac{1}{2}\,a_0 \cos n\theta\,d\theta + a_m \sum_{m=1}^{\infty}\int_0^{2\pi}\cos m\theta \cos n\theta\,d\theta + b_m \sum_{m=1}^{\infty}\int_0^{2\pi}\sin m\theta \cos n\theta\,d\theta$$

Substitute the results of the first part $\int_0^{2\pi}\cos m\theta \cos n\theta$ = (0, where m ≠ n, and π where m = n). The integral lies under the sum, which goes to infinity. There can be only one instance where m = n. Therefore the second term becomes $a_m\pi$.

$$= \frac{1}{\pi}\Big[\int_0^{2\pi}\frac{1}{2}\, a_0 \cos n\theta \, d\theta + a_m\pi + b_m\sum_{m=1}^{\infty}\int_0^{2\pi}\sin m\theta \cos n\theta \, d\theta]$$

$$=$$

$$\frac{1}{\pi}\Big[\frac{1}{2}\, a_0 \sin n\theta \,\Big|_0^{2\pi} + a_m\,\pi + b_m \sum_{m=1}^{\infty}\big(\frac{1}{m^2-n^2}\,(-n\sin(2\,\pi\, m)\sin(2\,\pi\, n) + m\,(-\cos(2\,\pi\, m))\cos(2\,\pi\, n) + m)\big)\Big]$$

$$= \frac{1}{\pi}[0 + a_m\pi + b_m\sum_{m=1}^{\infty}\frac{1}{m^2-n^2}(-n\sin(2\,\pi \mathrm{m})\sin(2\,\pi \mathrm{m}) \;+\; m\,(-\cos(2\,\pi\, m))\cos(2\,\pi\, n) + m)]$$

By inspection, we can see that the first term must vanish. The second term will go to a_m. In the numerator of the third term, the sine product vanishes and the cosine product becomes (–1), so the numerator becomes (m – m) = 0.

$$= \frac{1}{\pi}[0 + a_m\,\pi + 0] = a_m$$

Since m = n, $a_m = a_n$, so one can write $\frac{1}{\pi}\int_0^{2\pi}\mathcal{F}(\theta)\cos n\theta\, d\theta = a_n$. Then we have the first part of the validation of (19)*.

Now repeat for b_n. Substitute for $\mathcal{F}(\theta)$ in the equation for b_n and multiply through by sin nθ.

$$b_n = \frac{1}{\pi}\int_0^{2\pi}\Big[\frac{1}{2}\, b_0 \sin n\theta\, d\theta + \sum_{m=1}^{\infty}\,[a_m \cos m\theta \sin n\theta + b_m \sin m\theta \sin n\theta]\, d\theta$$

Rearrange the sum, integrals, and coefficients.

$$b_n \;= \frac{1}{\pi}[\int_0^{2\pi}\frac{1}{2}\, b_0 \sin n\theta\, d\theta + a_m \sum_{m=1}^{\infty}\int_0^{2\pi}\cos m\theta \sin n\theta\, d\theta + b_m \sum_{m=1}^{\infty}\int_0^{2\pi}\sin m\theta \sin n\theta\, d\theta]$$

Our previous work shows that the first two terms vanish. To obtain the third term, substitute the results of the first part $\int_0^{2\pi}\sin m\theta \sin n\theta$ = (0, where m ≠ n and π where m = n). The integral lies under the sum, which goes to infinity. There can be only one instance where m = n. Therefore the third term becomes $b_m\pi$ and

$b_n = \frac{1}{\pi}[0 \;+ 0 \;+\; b_m\,\pi] = b_m$, and we know that m = n, so one can write $\frac{1}{\pi}\int_0^{2\pi}\mathcal{F}(\theta)\sin n\theta\, d\theta = b_n$.

This completes the validation of (19)*. When $\mathcal{F}(\theta)$ is substituted into the coefficient equations, the result is the coefficients themselves.

There is a different, but entertaining derivation of the coefficient equations in T. W. Körner, *Fourier Analysis*. Cambridge, 1988, pp. 28-31.