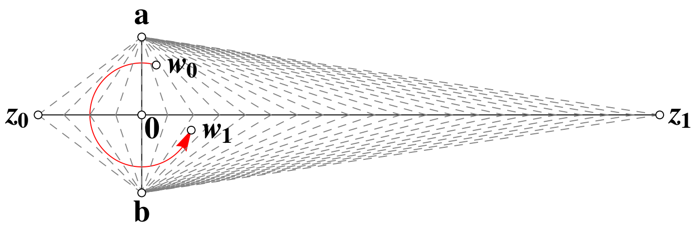
Needham, Chapter 2, Exercise 2, p. 111.
G. Palmer, May 19, 2020, 1:20 pm PST

Out[813]=



2. Consider the complex mapping z ⟶ w = (z-a)/(z-b). Show geometrically that if we apply this mapping to the perpendicular bisector of the line-segment joining a and b, then the image is the unit circle. In greater detail, describe the motion of w round this circle as z travels along the line at constant speed.

We can draw a and b on the y axis and use the real axis for the perpendicular bisector. Since the real axis is the perpendicular bisector of ab, z lies on the real axis, a0z and b0z are congruent triangles reversed in orientation, and |z-a| = |z-b| for all z. Let $\theta = \angle$azO. By inspection, we can see that as z approaches the origin from the negative direction, θ goes from 0 in the limit to $\pi/2$. As z continues in the positive direction, then θ goes from $\pi/2$ to π in the limit. Hence, the θ goes from 0 to π in a traversal from -∞ to ∞. The lines from above and below the real line meeting at z are of equal length. Below the real line the angles are $(-\theta)$. If we let (z-a) = $-re^{i\theta}$ and (z-b) = $-re^{-i\phi}$, then *(z-a)/(z-b)* = $e^{i2\theta}$, which describes a unit circle as z traverses from -∞ to ∞.

As z travels along the line at constant speed, θ changes most rapidly near the origin and most slowly as z goes to ±∞. The direction of traversal of w is counter clockwise, beginning at 1 for $z_0 \longrightarrow -\infty$, and ending at 2π for z ⟶ ∞.

Vasco has a more elegant approach to this. One simply needs to recognize that the two triangles az0 and bz0 are congruent, so that |z-a| = |z-b|. Then |w| = $\left|\frac{z-a}{z-b}\right| = \frac{|z-a|}{|z-b|}$ = 1. |w| = 1 is an equation for the unit circle.