Needham, Chapter 2, Exercise 19, pp. 116.
G. Palmer, August 31, 2020, 6 pm PST

19 Show that the ratio test cannot be used to find the radius of convergence of the power series (18) on p. 77. Use the root test to confirm that $R = \sqrt{1+k^2}$.

The power series (18)

$$\frac{1}{1+x^2} = \sum_{j=0}^{\infty} \left[\frac{\sin(j+1)\,\phi}{\left(\sqrt{1+k^2}\right)^{j+1}} \right] X^j$$

$\sqrt{1+k^2}$ is a constant, where k represents a centre of expansion.

Apply the ratio test $R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|$ substituting j for n.

$$\lim_{j\to\infty} \left| \frac{\frac{\sin(j+1)\,\phi}{\left(\sqrt{1+k^2}\right)^{j+1}}}{\frac{\sin(j+2)\,\phi}{\left(\sqrt{1+k^2}\right)^{j+2}}} \right| = \lim_{j\to\infty} \left| \frac{\left(\sqrt{1+k^2}\right)\sin(j+1)\,\phi}{\sin(j+2)\,\phi} \right|$$

In $\sqrt{1+k^2}$ Sin[(j+1)ϕ]/Sin[(j+2)ϕ], we find undefined points for all values of j. The points occur at (j+2)ϕ = nPi or ϕ = nPi/(j+2). Hence, the ratio test does not work for (18).

We try the root test $R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|c_n|}}$.

$$R = \lim_{j\to\infty} \frac{1}{\sqrt[j]{\left| \frac{\sin(j+1)\,\phi}{\sqrt{1+k^2}^{\,j+1}} \right|}} = \lim_{j\to\infty} \frac{1}{\sqrt[j]{\frac{|\sin(j+1)\,\phi|}{\left|\sqrt{1+k^2}^{\,j+1}\right|}}} = \lim_{j\to\infty} \frac{1}{\frac{|\sin(j+1)\,\phi|^{1/j}}{\left|\sqrt{1+k^2}^{\,j+1}\right|^{1/j}}}$$

$$= \lim_{j\to\infty} \frac{\left|\sqrt{1+k^2}^{\,j+1}\right|^{1/j}}{|\sin(j+1)\,\phi|^{1/j}} = \lim_{j\to\infty} \frac{\left|\sqrt{1+k^2}^{\,\frac{j+1}{j}}\right|}{|\sin(j+1)\,\phi|^{1/j}} = \frac{\sqrt{1+k^2}}{|\sin(j+1)\,\phi|^{0}}$$

$$= \sqrt{1+k^2}$$ This is what we wanted.