Needham, Chapter 2, Exercise 18, p. 116, parts (v), (vi).
G. Palmer, August 31, 2020, 10:10 am PST

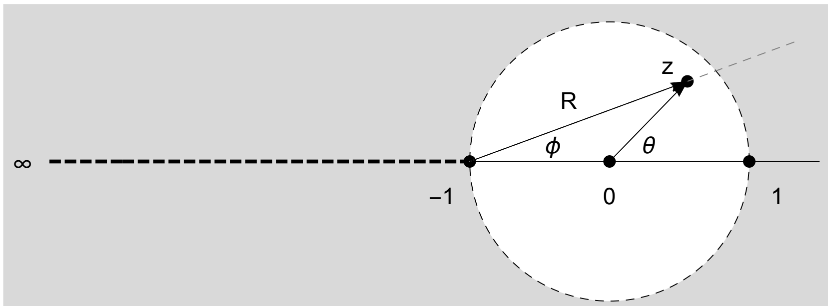

Figure 1a. Domain of z in convergent binomial series is unit disc, |z| < 1,
series is 0 branch of multifunction $(1+z)^{\frac{1}{q}}$,
branch cut of multifunction −1 to −∞

Out[184]=

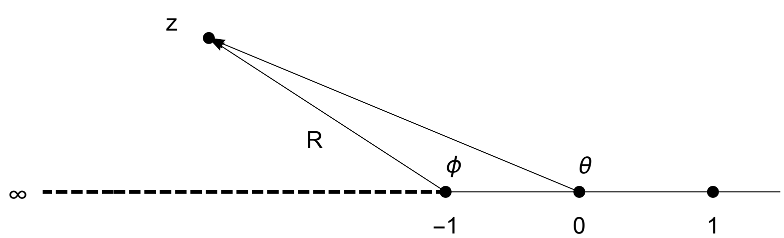

Figure 1b. Domain of z about point −1
in any of 1 to (q−1) branches of multifunction $(1+z)^{\frac{1}{q}}$
is the complex plane,
branch cut of multifunction −1 to −∞

(v) Use (43) to show that if q is an integer then $\left[B\left(z, \frac{1}{q}\right)\right]^q = (1+z)$. *Deduce that* $B(z, \frac{1}{q})$ *is the* princi-pal *branch of* $(1+z)^{\frac{1}{q}}$.

From (43), we can write, by substituting multiplication by q for the addition of q + q + q ... q times.

$$\left[B\left(z, \frac{1}{q}\right)\right]^q = B\left(z, q \cdot \frac{1}{q}\right) = (1+z)$$

$B(z, \frac{1}{q})$ is the name of the binomial power series. If we extended the Binomial Theorem to the complex plane and rational powers, the series is the RHS of

$$f(z) = (1+z)^{\frac{1}{q}} = 1 + \frac{1}{q} z + \frac{\frac{1}{q}\left(\frac{1}{q}-1\right)}{2!} z^2 + \frac{\frac{1}{q}\left(\frac{1}{q}-1\right)\left(\frac{1}{q}-2\right)}{3!} z^3 + \frac{\frac{1}{q}\left(\frac{1}{q}-1\right)\left(\frac{1}{q}-2\right)\left(\frac{1}{q}-3\right)}{4!} z^4 + \cdots$$

$$= 1 + \frac{1}{q} z + \frac{1-q}{2!\, q^2} z^2 + \frac{2q^2-3q+1}{3!\, q^3} z^3 + \frac{-6q^3+11q^2-6q+1}{4!\, q^4} z^4 + \cdots \qquad (2)$$

which is a single-valued function where $(1+z)^{\frac{1}{q}}$ on the LHS is a q-valued function. Needham presents a description of the principal branch of $\sqrt[3]{z}$ on p. 94. We can adapt it and the discussion on pp. 95-96 to show that B(z, $\frac{1}{q}$) is the principal branch of $(1+z)^{\frac{1}{q}}$. Consider a z beginning at a point **p** on the complex plane and traversing on a closed loop around a branch point –1 in the counterclockwise direction at some possibly variable radius from the branch point. Assume also that z encircles the branch point q times before returning to **p**. If we make a branch cut on the negative x axis from 0 to $-\infty$, the principal branch is the first (0^{th}) angular traversal where $-\pi < \theta \leq \pi$ is the principal value of the argument. The convergent series is defined only within the unit disc (white in Figure 1a). With this constraint on θ and z, let z = $re^{i\theta}$. Then $[1+z] = (1 + [re^{i\pi\theta}])$. We can write

$$[(1+z)^{\frac{1}{q}}] = (1 + [re^{i\pi\theta}])^{\frac{1}{q}}$$

Now let $[(1+z)^{\frac{1}{q}}] = R^{\frac{1}{q}}[e^{i\frac{1}{q}\pi\phi}]$, where R = $|1+z|$ and ϕ is arg$(1+z)$ where z = $[re^{i\pi\theta}]$. Since the series is convergent only when z is within the unit disc, it must be the case that $-\pi/2 < \phi \leq \pi/2$. When z = 0, both the binomial $B\left(z, \frac{1}{q}\right)$, which is a single-valued function, and the principal branch $\left[(1+z)^{\frac{1}{q}}\right]$ of the multifunction occur on the real line with value 1, showing that $B(z, \frac{1}{q})$ is equivalent to the principal branch. Subsequent branches [1 to (q-1)] of the multifunction can be written $R^{\frac{1}{q}}\, e^{i\frac{2}{q}\pi\phi}, R^{\frac{1}{q}}\, e^{i\frac{3}{q}\pi\phi}, \cdots, R^{\frac{1}{q}}\, e^{i\frac{q-1}{q}\pi\phi}$.

(vi) Finally, show that if p and q are integers, then B(z, $\frac{p}{q}$) is indeed the principal *branch of* $(1+z)^{\frac{p}{q}}$.

Consider a z beginning at a point **p** (not the same as the point p in (p/q)) on the complex plane and traversing on a closed loop around a branch point –1 in the counterclockwise direction at some possibly variable radius from the branch point. Assume that z encircles the branch point q times before returning to **p**. If we make a branch cut on the negative x axis, the principal branch is the first angular traversal where $-\pi < \theta \leq \pi$ is the principal value of the argument. Proceeding as in (v), we can write

$$[(1+z)^{\frac{p}{q}}] = R^{\frac{p}{q}}\left[e^{i\frac{p}{q}\pi\phi}\right]$$

This is the principal branch of a q-valued multifunction. B(z, $\frac{p}{q}$) is the name of the single-valued binomial power series, which is the RHS of

$$f(z) = (1+z)^{\frac{p}{q}} = 1 + \frac{p}{q}z + \frac{\frac{p}{q}\left(\frac{p}{q}-1\right)}{2!}z^2 + \frac{\frac{p}{q}\left(\frac{p}{q}-1\right)\left(\frac{p}{q}-2\right)}{3!}z^3 + \frac{\frac{p}{q}\left(\frac{p}{q}-1\right)\left(\frac{p}{q}-2\right)\left(\frac{p}{q}-3\right)}{4!}z^4 + \cdots$$

$$= 1 + \frac{p}{q} z + \frac{p(p-q)}{2!\, q^2} z^2 + \frac{p\left(p^2-3\, p\, q+2\, q^2\right)}{3!\, q^3} z^3 + \frac{p\left(p^3-6\, p^2\, q+11\, p\, q^2-6\, q^3\right)}{4!\, q^4} z^4 + \cdots$$

When z = 0, both the binomial $B(z, \frac{p}{q})$ and the principal branch $R^{\frac{p}{q}}\left[e^{i\, \frac{p}{q}\, \pi\, \phi}\right]$ occur on the real line with value 1, showing that $B(z, \frac{p}{q})$ is equivalent to the principal branch. Branches n = 1..(q-1) can be written $R^{\frac{p}{q}}\left[e^{i\, \frac{p}{q}\, \mathrm{n}\pi\, \phi}\right]$.