Needham, Chapter 2, Exercise 18, p. 116, parts (iii), (iv).
G. Palmer, August 30, 2020, 11 am PST

(iii) If m and n are positive integers, then show that

$$B(z, n)\, B(z, m) = B(z, n+m)$$

and deduce that $C_r(n,\ m) = \binom{n+m}{r}$. *But* $C_r(n, m)$ *and* $\binom{n+m}{r}$ *are simply polynomials in n and m, and so the fact that they agree at infinitely many values of m and n [positive integers] implies that they must be equal for all real values of m and n. Thus the key formula (43) is valid for all real values of m and n.*

$$\mathrm{B(z, n)\ B(z, m)} = (1 + z)^n\, (1 + z)^m = (1 + z)^{n+m} = \mathrm{B(z, n+m)}$$

This establishes (43).

Since $\mathrm{B(z, n)} \equiv \sum_{r=0}^{\infty} \binom{n}{r} z^r$, we can write

$$\mathrm{B(z, n + m)} \equiv \sum_{r=0}^{\infty} \binom{n+m}{r} z^r.$$

Then, by (ii),

$$\mathrm{B(z, n + m) = B(z, n)\ B(z, m)} = \sum_{r=0}^{\infty} C_r(n, m)\, z^r \text{ where } C_r(n, m) = \sum_{j=0}^{r} \binom{n}{j}\binom{m}{r-j}.$$

Hence,

$$C_r(n,\ m) = \binom{n+m}{r}.$$

(iv) By substituting n = –m in (43), deduce the Binomial Theorem for negative integer values of n.

$$\mathrm{B(z, n)\ B(z, m) = B(z, n + m)} \qquad (43)$$

$$\mathrm{B(z, –m)\ B(z, m) = B(z, 0)} = (1 + z)^0 = 1$$

From (ii) we have

$$\mathrm{B(z, n)\ B(z, m)} = \sum_{r=0}^{\infty} C_r(n, m)\, z^r \text{ where } C_r(n, m) = \sum_{j=0}^{r} \binom{n}{j}\binom{m}{r-j}.$$

Then

$$B(z, -m)\, B(z, m) = \sum_{r=0}^{\infty} \sum_{j=0}^{r} \binom{-m}{j} \binom{m}{r-j} z^r = 1 \qquad (1)$$

From this I think we can deduce that

$$B(z, -m) = \sum_{j=0}^{r} \binom{-m}{j} \equiv \sum_{j=0}^{r} \frac{-m(-m-1)(-m-2)\ldots(-m-j+1)}{j!} z^j, \text{ where } j \to r \text{ and } r \to \infty$$

and letting $r \to \infty$,

$$B(z, -m) = \sum_{j=0}^{\infty} \binom{-m}{j} \equiv \sum_{j=0}^{\infty} \frac{-m(-m-1)(-m-2)\ldots(-m-j+1)}{j!} z^j$$

Substitute $n = -m$ to obtain the general Binomial Theorem (14, p. 73), which works for all real numbers when z is within the unit disc.