Needham, Chapter 2, Exercise 18, p. 116.
G. Palmer, August 30, 2020, 11 am PST

18. The basic idea of the following argument is due to Euler. Initially, let n be any real (possibly irrational) number and define

$$B(z, n) \equiv \sum_{r=0}^{\infty} \binom{n}{r} z^r \text{ where } \binom{n}{r} \equiv \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$$

and $\binom{n}{0} \equiv 1$*. We know from elementary algebra that if n is a positive integer then* $B(z, n) = (1+z)^n$*. To establish the Binomial Theorem (14) for rational powers, we must show that if p and q are integers then* $B(z, \frac{p}{q})$ *is the* principal *branch of* $(1+z)^{\frac{p}{q}}$.

(i) With a fixed value of n, use the ratio test to show that B(z, n) converges in the unit disc, $|z| < 1$.

The ratio test shown on p. 74 uses first j and then n to index the coefficients. Note that they take the same index letter as the power of z. In our problem statement, the power of z is indexed with the letter r, so that is what we use here.

$$\lim_{r\to\infty} \left|\frac{C_r}{C_{r+1}}\right| = \lim_{r\to\infty} \left|\frac{\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}}{\frac{n(n-1)(n-2)\ldots(n-(r+1)+1)}{(r+1)!}}\right| = \lim_{r\to\infty}\left|\frac{n(n-1)(n-2)\ldots(n-r+1)(r+1)!}{n(n-1)(n-2)\ldots(n-r)\,r!}\right|$$

$$= \lim_{r\to\infty}\left|\frac{r+1}{n-r}\right| = 1$$

As r --> ∞, 1 and n become insignificant. Since the quotient never actually quite reaches 1, we can say that B(z, n) converges in the unit disc, $|z| < 1$.

(ii) By multiplying the two power series, deduce that

$$B(z, n)\, B(z, m) = \sum_{r=0}^{\infty} C_r(n, m)\, z^r \text{ where } C_r(n, m) = \sum_{j=0}^{r} \binom{n}{j}\binom{m}{r-j}.$$

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$$

$$B(z, n)\, B(z, m) = \sum_{r=0}^{\infty} \binom{n}{r} z^r \sum_{r=0}^{\infty} \binom{m}{r} z^r$$

$$= \binom{n}{0} + \binom{n}{1} z + \binom{n}{2} z^2 + \cdots + \binom{n}{\infty} z^{\infty} \qquad \left(* \binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} *\right)$$

$$\sum_{r=0}^{\infty} \binom{m}{r} z^r = \binom{m}{0} + \binom{m}{1} z + \binom{m}{2} z^2 + \cdots + \binom{m}{\infty} z^{\infty}$$

Let us assume that we will be multiplying with term 0 of the first series times the second series. Then we can write

$$\sum_{r=0}^{\infty}\binom{n}{r}z^r = \sum_{j=0}^{r}\left[\binom{n}{j}z^j \sum_{r=0}^{\infty}\binom{m}{r}z^r\right]$$

Multiply the two series

$$= \binom{n}{0}z^0 \sum_{r=0}^{\infty}\binom{m}{r}z^r + \binom{n}{1}z^1\sum_{r=0}^{\infty}\binom{m}{r}z^r + \binom{n}{2}z^2\sum_{r=0}^{\infty}\binom{m}{r}z^r + \cdots$$

$$= \binom{n}{0}\binom{m}{0}z^0z^0 + \binom{n}{0}\binom{m}{1}z^0\, z^1 + \binom{n}{0}\binom{m}{2}z^0z^2 + \cdots$$

$$+ \binom{n}{1}\binom{m}{0}z^1z^0 + \binom{n}{1}\binom{m}{1}z^1\, z^1 + \binom{n}{1}\binom{m}{2}z^1\, z^2 + \cdots$$

Consolidate the powers of z and align the coefficients of each power in columns.

$$\binom{n}{0}\binom{m}{0} + \binom{n}{0}\binom{m}{1}\mathrm{z} + \binom{n}{0}\binom{m}{2}\, z^2 + \binom{n}{0}\binom{m}{3}\, z^3 + \binom{n}{0}\binom{m}{4}\, z^4 + \cdots + \binom{n}{0}\binom{m}{\infty}z^\infty$$

$$+ \binom{n}{1}\binom{m}{0}\mathrm{z} + \binom{n}{1}\binom{m}{1}z^2 + \binom{n}{1}\binom{m}{2}\, z^3 + \binom{n}{1}\binom{m}{3}\, z^4 + \binom{n}{1}\binom{m}{4}\, z^5 + \cdots + \binom{n}{1}\binom{m}{\infty}$$

$$z^\infty$$

$$+ \binom{n}{2}\binom{m}{0}z^2 + \binom{n}{2}\binom{m}{1}z^3 + \binom{n}{2}\binom{m}{2}\, z^4 + \binom{n}{2}\binom{m}{3}\, z^5 + \binom{n}{2}\binom{m}{4}\, z^6$$

$$+ \cdots + \binom{n}{2}\binom{m}{\infty}z^\infty$$

$$+ \binom{n}{3}\binom{m}{0}z^3 + \binom{n}{3}\binom{m}{1}z^4 + \ldots$$

Sum the columns.

$$= \binom{n}{0}\binom{m}{0} + \left[\binom{n}{0}\binom{m}{1} + \binom{n}{1}\binom{m}{0}\right]\mathrm{z} + \left[\binom{n}{0}\binom{m}{2} + \binom{n}{1}\binom{m}{1} + \binom{n}{2}\binom{m}{0}\right]z^2 + \left[\binom{n}{0}\binom{m}{3} + \binom{n}{1}\binom{m}{2} + \binom{n}{2}\binom{m}{1} + \binom{n}{3}\binom{n}{3}\binom{m}{0}\right]z^3 + \cdots$$

Read value of r as the power of z. Read value of j from $\binom{n}{j}$.

$$= \binom{n}{0}\binom{m}{r-0} z^0 + \left[\binom{n}{0}\binom{m}{r-0} + \binom{n}{1}\binom{m}{r-1}\right] z^1 + \left[\binom{n}{0}\binom{m}{r-0} + \binom{n}{1}\binom{m}{r-1} + \binom{n}{2}\binom{m}{r-2}\right] z^2$$

$$+ \left[\binom{n}{0}\binom{m}{r-0} + \binom{n}{1}\binom{m}{r-1} + \binom{n}{2}\binom{m}{r-2} + \binom{n}{3}\binom{m}{r-3}\right] z^3 + \cdots$$

$= \sum_{r=0}^{\infty} C_r(n, m)\, z^r$ *where* $C_r(n, m) = \sum_{j=0}^{r} \binom{n}{j}\binom{m}{r-j}$ for all real numbers n and m.