Needham, Chapter 2, Exercise 17, p. 115
G. Palmer, August 10, 2020, 4:30 pm PST

17 Here is an inductive approach to the result of the previous exercise.

I was getting nowhere with this answer, so I read Vasco's answer and tried to answer it using his approach.

(i) Write down the first few rows of Pascal's triangle and circle the numbers $\binom{5}{3}, \binom{4}{2}, \binom{3}{1}, \binom{2}{0}$. *Check that the sum of these numbers is* $\binom{6}{3}$. *Explain this.*

	n
1	0
1 1	1
1 2 **1**	2
1 **3** **3** 1	3
1 4 **6** 4 1	4
1 5 **10** **10** 5 1	5

The numbers to be circled are rendered in bold. They arrange into two symmetrical diagonal lines in row order from top to bottom 1, 3, 6, 10. The sum of the numbers in each diagonal is 20. Let n be the row number and let r be the position within each row. Both n and r begin the count at 0. The apex number 1 is at row 0, position 0. We can designate this number as $\binom{n}{r} = \binom{0}{0}$. The 0$^{\text{th}}$ number in the second row can be written $\binom{n}{r} = \binom{2}{0}$. At the 1 position, $\binom{n}{r} = \binom{2}{1}$. We have three ways of calculating numbers by row and position.

(1) Each number in Pascal's triangle is the sum of the two proximate numbers in the immediately preceding row. If there is only one proximate number in the preceding row, the other is taken to be 0.

(2) We can calculate the numbers using the row and position in the formula for the Binomial Theorem from page 50, Exercise 32. Thus, the numbers in rows 0 to 2 are $\left[\binom{0}{0} = 1\right]$ for row 0, $\left[\binom{1}{0} = 1,\ \binom{1}{1} = 1\right]$ for row 1, and $\left[\binom{2}{0} = 1,\ \binom{2}{1} = 2, \binom{2}{2} = 1\right]$ for row 2.

(3) We can observe that all the 0 positions equal $\binom{0}{0} = 1$ and all the final positions equal $\binom{n}{n} = 1$, which means the left and right sides of the triangle consist entirely of 1's. Just inside the 0 and final positions, but actually beginning in row 1, there are two diagonals in which each succeeding number increments the previous by 1. In the diagonal row of interest in this exercise, with numbers 1, 3, 6, 10, we can observe that the *difference* between two successive numbers increases by 1 in each

successive row. The differences are 2, 3, 4, 5, ⋯. The next number in the row would be 10 + 5 = 15, which can also be obtained by using the general method of adding the two proximate numbers in the preceding row.

Check that the sum of these numbers is $\binom{6}{3}$.

$$\binom{5}{3} = \frac{5!}{2! \times 3!} = 10, \quad \binom{4}{2} = \frac{4!}{2! \times 2!} = 6, \quad \binom{3}{1} = 3, \quad \binom{2}{0} = 1$$

$$10+6+3+1 = 20 = \frac{6*5*4*3*2}{(6-3)!\,3!} = \binom{6}{3}$$

The numbers 10, 6, 3, 1 are the coefficients of the power series of $\left((1-z)^{-1}\right)^3 = (1-z)^{-3}$.

$$P(z) = 1 + 3z + 6\,z^2 + 10\,z^3 + \ldots$$

How could we predict this from the triangle? Perhaps because the diagonal coefficient series begins in row 2. If so, then a series beginning in row 3 would consist of the coefficients of $\left((1-z)^{-1}\right)^4 = (1-z)^{-4}$. *Mathematica* shows this to be the case by calculating the diagonal series from row 3, position 0 to row 6, position 3, in three different ways: (a) by finding the series of the 4th power of the function; (b) by finding the series of $(1-z)^{-1}$ and raising it to the power (–4); and (c) by taking the series to the (–n) power and substituting n = 4.

```
Clear["Global`*"];
a = Series[((1 - z) ^ (-1)) ^4, {z, 0, 3}];
b = Series[(1 - z) ^ (-1), {z, 0, 3}] ^4;
Print["a: ", a];
Print["b: ", b];
c = Series[(1 - z) ^ (- n), {z, 0, 3}];
Print["c: ", c];
n = 4;
Print["c: ", c];
```

a: $1 + 4\,z + 10\,z^2 + 20\,z^3 + O(z^4)$

b: $1 + 4\,z + 10\,z^2 + 20\,z^3 + O(z^4)$

c: $1 + n\,z + \frac{1}{2}\left(n^2 + n\right) z^2 + \frac{1}{6}\left(n^3 + 3\,n^2 + 2\,n\right) z^3 + O(z^4)$

c: $1 + 4\,z + 10\,z^2 + 20\,z^3 + O(z^4)$

(ii) Generalize your argument to show that

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-2}{r-1} + \binom{n-3}{r-2} + \ldots + 1.$$

Begin with Pascal's triangle as presented above and add one more row to include the number 20 at position 3.

	n
1	0
1 1	1
1 2 **1**	2
1 **3** **3** 1	3
1 4 **6** 4 1	4
1 5 **10** **10** 5 1	5
1 6 15 20 15 6 1	6

Let row 6 be n. We can see that 20 is in position 3 of row 6. We can generalize this number to $\binom{n}{r}$. We use the formula that 20 is the sum of 10 plus the preceding numbers in the leftward slanting boldface diagonal. The number 10 that we want is $\binom{5}{3}$. We can generalize this to $\binom{n-1}{r}$. The next in the sum is 6 = $\binom{4}{2}$, which generalizes to $\binom{n-2}{r-1}$. Then 3 = $\binom{3}{1}$, which generalizes to $\binom{n-3}{r-2}$, and so on. Finally 1 = $\binom{2}{2}$. Adding these, we obtain

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-2}{r-1} + \binom{n-3}{r-2} + \binom{n-4}{r-3} = 20$$

The example is confined to rows 0 to 6, but if the number of rows is indefinite, we can write

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-2}{r-1} + \binom{n-3}{r-2} + \cdots + 1$$

(iii) Assume that $(1-z)^{-M} = \sum_{r=0}^{\infty} \binom{M+r-1}{r} z^r$ *holds for some positive integer M. Now, multiply this series by the geometric series for* $(1-z)^{-1}$ *to find* $(1-z)^{-(M+1)}$*. Deduce that the binomial series is valid for all negative integer powers.*

$$(1-z)^{-(M+1)} = \sum_{r=0}^{\infty} \binom{M+r-1}{r} z^r \bullet [1 + z + z^2 + z^3 + z^4 + \cdots]$$

Expand the series for $(1-z)^{-M}$:

$$\sum_{r=0}^{\infty} \binom{M+r-1}{r} z^r = \binom{M+0-1}{0} z^0 + \binom{M+1-1}{1} z^1 + \binom{M+2-1}{2} z^2 + \cdots$$

$$= \binom{M-1}{0} z^0 + \binom{M}{1} z^1 + \binom{M+1}{2} z^2 + \binom{M+2}{3} z^3 + \binom{M+3}{4} z^4 + \cdots$$

Multiply the –M power series by the –1 power series.

$$[1 + \binom{M}{1} z + \binom{M+1}{2} z^2 + \binom{M+2}{3} z^3 + \binom{M+3}{4} z^4 + \cdots] \bullet [1 + z + z^2 + z^3 + z^4 + \cdots]$$

Add the rows.

$$= 1 \quad + z \quad + z^2 \quad + z^3 \quad + z^4 + \cdots$$

$$+ \binom{M}{1} z + \binom{M}{1} z^2 + \binom{M}{1} z^3 + \binom{M}{1} z^4 + \binom{M}{1} z^5 + \cdots$$

$$+ \binom{M+1}{2} z^2 + \binom{M+1}{2} z^3 + \binom{M+1}{2} z^4 + \binom{M+1}{2} z^5 + \cdots$$

$$+ \binom{M+2}{3} z^3 + \binom{M+2}{3} z^4 + \binom{M+2}{3} z^5 + \cdots$$

Collect like powers.

$$1 + [1 + \binom{M}{1}] z + [1 + \binom{M}{1} + \binom{M+1}{2}] z^2 + [1 + \binom{M}{1} + \binom{M+1}{2} + \binom{M+2}{3}] z^3 + \cdots$$

From part (ii), we know that

$$[1 + \binom{M}{1}] = \binom{M+1}{1}$$

$$[1 + \binom{M}{1} + \binom{M+1}{2}] = \binom{M+2}{2}$$

...

Proceeding in this way

$$(1-z)^{-(M+1)} = 1 + \binom{M+1}{1} z + \binom{M+2}{2} z^2 + \binom{M+3}{3} z^3 + \cdots = \sum_{r=0}^{\infty} \binom{M+r}{r} z^r$$

For a proof by induction, we need a base case. Let r = 0. Then $\sum_{r=0}^{\infty} \binom{M+0}{0} z^0 = \frac{M!}{(M-0)!\,0!} = 1$. This is correct for the 0 position in any row. We assumed that $(1-z)^{-M} = \sum_{r=0}^{\infty} \binom{M+r-1}{r} z^r$ holds for some positive integer M. Since

$$\binom{M+r}{r} = \binom{(M\ +\ 1) + r - 1}{r},$$

we have also shown that it holds for M + 1, which means that the Binomial Theorem works for all negative integer powers of (1-z).