Needham, Chapter 2, Exercise 16, p. 115
G. Palmer, August 2, 2020, 12:20 pm PST

16. Our aim is to give a combinatorial explanation of the Binomial Theorem (14) for all negative integer values of n. The simple yet crucial first step is to write n = –m and to change z to –z. Check that the desired result (14) now takes the form $(1-z)^{-m} = \sum_{r=0}^{\infty} c_r z^r$*, where* c_r *is the binomial coefficient*

$$c_r = \binom{m+r-1}{r}$$

From p. 73, we have the binomial theorem expressed as

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{2!} z^2 + \frac{n(n-1)(n-2)}{3!} z^3 + \frac{n(n-1)(n-2)(n-3)}{4!} z^4 + ... \quad (14)$$

substitute n = –m and z = –z

$$(1-z)^{-m} = 1 + (-m)(-z) + \frac{-m(-m-1)}{2!} z^2 - \frac{-m(-m-1)(-m-2)}{3!} z^3 + \frac{-m(-m-1)(-m-2)(-m-3)}{4!} z^4 + ...$$

$$= 1 + mz + \frac{m(m+1)}{2!} z^2 + \frac{m(m+1)(m+2)}{3!} z^3 + \frac{m(m+1)(m+2)(m+3)}{4!} z^4 + ...$$

This can be written

$$\sum_{r=0}^{m} \frac{(m+r-1)!}{(m-1)!\, r!} z^r = \sum_{r=0}^{m} \frac{(m+r-1)!}{((m+r-1)-r)!\, r!} z^r \text{ , because } \binom{n}{k} = \frac{n!}{(n-k)!\, k!}$$

$$= \sum_{r=0}^{m} \binom{m+r-1}{r} z^r$$

$$= \sum_{r=0}^{\infty} c_r z^r$$

Hence,

$$(1-z)^{-m} = \sum_{r=0}^{m} \binom{m+r-1}{r} z^r = \sum_{r=0}^{\infty} c_r z^r \text{, where } c_r = \binom{m+r-1}{r}$$

[Note that this says that the coefficients c_r *are obtained by reading Pascal's triangle diagonally, instead of horizontally.] To begin to understand this, consider the special case m = 3. Using the geometric series for* $(1-z)^{-1}$*, we may express* $(1-z)^{-3}$ *as*

$$[1 + z + z^2 + z^3 + \cdots] \bullet [1 + z + z^2 + z^3 + \cdots] \bullet [1 + z + z^2 + z^3 + \cdots],$$

where • simply denotes multiplication. Suppose we want the coefficient c_9 *of* z^9*. One way to get* z^9 *is* *to take z^3 from the first bracket, z^4 from the second and z^2 from the third.*

(i) Write this way of obtaining z^9 as the sequence zzz•zzzz•zz of 9 z's and 2 •'s, where the latter keep track of which power of z came from which bracket. [...] Explain why c_9 is the number if distinguishable rearrangements of this sequence of 11 symbols. Be sure to address the meaning of sequences in which a • comes first, last, or is adjacent to the other •.

Examples:

z•z•zzzzzzz
z•zz•zzzzzz
z•zzz•zzzzz
...

zz•z•zzzzzz
zz•zz•zzzzz
...

zzz•z•zzzzz
zzz•zz•zzzz
...

c_9 is the number of distinguishable rearrangements of the z• sequence because each muliplication of three factors (one from each bracket) that produces z^9 adds 1 to c_9. The distinguishable rearrangements represent all possible arrangements of $z^a\, z^b\, z^c$, where a, b, and c range from 0 to 9, a+b+c = 9, and z^a, z^b, z^c come from separate brackets, i.e. $[]_a$, $[]_b$, $[]_c$. Putting • first, last, or in the order •• is equivalent to z^0•...z..., ... z ... •z^0, or ... z ... •z^0 • ... z .... Vasco described this as z^0 is chosen from the first, third, or second bracket, respectively.

(ii) Deduce that c_9 = $\binom{11}{9}$, in agreement with (42).

$$c_r = \binom{m+r-1}{r} \qquad (42)$$

Here is the hard part. We need to find the total number of possible sequences and the total number of indistinguishable sequences. The total number of possible sequences is the **product** of the total number of indistinguishable sequences and the total number of distinguishable sequences (n_d). To find the total number of possible sequences of all 11 symbols ••zzzzzzzzz, we have 11 choices at the first pick, 10 at the second, 9 at the third, ..., so the total number of possible sequences is 11! To find the total number of indistinguishable choices, we have 9! choices of the z's and 2! choices of the bullets. Then the total number of indistinguishable choices is the product 9!2!.

$$11! = n_d\, 9!\, 2!$$

$$n_d = \frac{11!}{9!\ \ 2!} = \frac{n!}{(n-k)!\,k!} = \frac{11!}{(11-9)!\,9!} = \binom{m+r-1}{r} = \frac{(3+9-1)!}{(3+9-1-9)!\,9!} = \binom{11}{9} = c_9$$

and

r = 9 because we want c_r. Presumably, we could also write

$$n_d = \frac{11!}{9!\ \ 2!} = \frac{n!}{(n-k)!\,k!} = \frac{11!}{(11-2)!\,2!} = \binom{11}{2}$$

and

$$\binom{11}{2} = \binom{11}{9} = c_9$$

(iii) Generalize this argument and thereby deduce (42).

$(1-z)^{-m}$ can be written as $\left((1-z)^{-1}\right)^m$

$(1-z)^{-1} = [1 + z + z^2 + ...]$

$\left((1-z)^{-1}\right)^m = [1 + z + z^2 + ...] \bullet [1 + z + z^2 + ...] \bullet [1 + z + z^2 + ...]$.... where there are m polynomials in brackets and m–1 bullets. The coefficient c_r of any rth power of z in the product polynomial is the number of product terms with z^r. Using the representation of the product terms with z's and bullets, as in ...z...•...z...•...z..., we can calculate the number of possible arrangements as (m + r – 1) and the number of indistinguishable arrangements as (m–1)!r!. The number of distinguishable arrangements is

$$n_d = \frac{(m+r-1)!}{(m-1)!\,r!} = \binom{m+r-1}{r} = c_r,$$

where m = power of the polynomial and r is the power of the term for which a coefficient is needed. Hence,

$$(1-z)^{-m} = \sum_{r=0}^{\infty} \binom{m+r-1}{r} z^r$$