Neeham, Chapter 2, Exercise 15, p. 114.
G. Palmer, August 4, 2020, 5:45 pm PST

15. Give an example of a pair of origin-centered power series, say P(z) and Q(z), such that the disc of convergence for the product P(z)Q(z) is larger than either of the two discs of convergence for P(z) and Q(z). [Hint: think in terms of rational functions, such as $\left[z^2 \,/\, (5-z)^3\right]$*, which are known to be expressible as power series.]*

The greatest difficulty with this problem is deciding on good functions to use for the two rational functions. We are not sure what functions can be expressed as power series. We learn the term "rational function" refer to "any function which can be defined by a rational fraction, i.e. an algebraic fraction such that both the numerator and the denominator are polynomials" (Wikipedia).

Keeping it simple, we choose for a reason that will be come evident,

$$f(z) = \frac{z^2+2z}{1-z}, \quad g(z) = \frac{z-z^2}{(2+z)(3-z)}$$

We know by (15) on p. 75 that

If f(z) can be expressed as a power series centered at k, then the radius of convergence is the distance from k to the nearest singularity of f(z).

We can see that f(z) has a singularity at z = 1 and g(z) has singularities at z = –2 and z = 3. Hence, f(z) with the nearest singularity at 1 has R = 1. The nearest singularity in g(z) is –2, so g(z) has R = |–2| = 2.

Multiply g(z) f(z):

$$g(z)f(z) = \frac{z^2+2z}{(1-z)}\,\frac{z-z^2}{(2+z)(3-z)} = \frac{z(z+2)\,z(1-z)}{(1-z)(2+z)(3-z)} = \frac{z^2}{(3-z)}$$

We deduce that since both f and g can be expressed as power series centred at 0, their product can also be expressed as a power series centred at 0. Vasco made a simplifying move by ensuring that singularities at short distances would cancel out in the numerator and denominator. So we have done the same. The product has a radius of convergence of 3, which is larger than that of either f(z) or g(z).

Since we know that the product has a larger disc of approximation than either f or g, one might argue that it is sufficient write general descriptions of origin-centred power series f(z) = P(z) = $\sum_{n=0}^{\infty} c_n z^n$ and g(z) = Q(z) = $\sum_{m=0}^{\infty} c_m z^m$.

The exercise states *Give an example of a pair of origin-centered power series,* which one might interpret to mean that one should give the first three to five coefficients of each or a formula for c_n and c_m as Vasco did in his answer.

From p. 76, we know that we can write 1/(a-z) = $\sum_{j=0}^{\infty} \frac{Z^J}{(a-k)^{j+1}}$, where Z = (z–k) and |z–k| < |a–k|. In this exercise, k = 0 so we can write 1/(a–z) = $\sum_{j=0}^{\infty} \frac{Z^J}{a^{j+1}}$. Then, from above,

$$f(z) = \frac{z^2+2z}{1-z} = \frac{z(z+2)}{1-z}, \qquad g(z) = \frac{z-z^2}{(2+z)(3-z)} = \frac{z(1-z)}{(2+z)(3-z)}$$

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Apart[z (1 - z) / ((2 + z) (3 - z))]
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$$-\frac{6}{5(z+2)} + \frac{6}{5(z-3)} + 1$$

f(z) has the partial fraction [– (z + 3) + $\frac{3}{1-z}$].

$$P(z) = -3 - z + 3\sum_{j=0}^{\infty} \frac{z^j}{1^{j+1}} \quad \text{if and only if } |z| < 1$$

$$= -3 - z + 3\sum_{j=0}^{\infty} z^j$$

$$= -3 - z + 3\left(1 + z + \sum_{j=2}^{\infty} z^j\right)$$

$$= 2z + 3\sum_{j=2}^{\infty} z^j$$

g(z) has the partial fraction [$-\frac{6}{5(z+2)} + \frac{6}{5(z-3)} + 1$].

$$1 - \frac{6}{5(z+2)} + \frac{6}{5(z-3)} = 1 + \frac{6}{5((-2)-z)} - \frac{6}{5(3-z)} = 1 + \frac{6}{5}\left(\frac{1}{(-2)-z} - \frac{1}{(3-z)}\right)$$

$$Q(z) = 1 + \frac{6}{5}\left(\sum_{j=0}^{\infty} \frac{z^j}{(-2)^{j+1}} - \sum_{j=0}^{\infty} \frac{z^j}{(3)^{j+1}}\right) \quad \text{if and only if } |z| < 2$$

$$= 1 + \frac{6}{5}\left(-\frac{1}{2} - \frac{1}{3}\right) + \frac{6}{5}\sum_{j=1}^{\infty}\left(\frac{z^j}{(-2)^{j+1}} - \frac{z^j}{(3)^{j+1}}\right) \quad \text{extract 0th powers and reindex}$$

$$= \frac{6}{5}\sum_{j=1}^{\infty} \frac{(3)^{j+1} z^j - (-2)^{j+1} z^j}{(3)^{j+1}(-2)^{j+1}}$$

$$= \frac{6}{5}\sum_{j=1}^{\infty} \frac{\left[(3)^{j+1} - (-2)^{j+1}\right] z^j}{(-6)^{j+1}}$$

$$= \sum_{j=1}^{\infty} \frac{\left[(3)^{j+1} - (-2)^{j+1}\right] z^j}{5 \cdot (-6)^j}$$

This provides *an example of a pair of origin-centered power series, ... P(z) and Q(z), such that the* *disc of convergence for the product P(z)Q(z) is larger than either of the two discs of convergence for P(z) and Q(z).*

$$R_Q = \underset{j\to\infty}{\text{Lim}} \left| \frac{\left[(3)^{j+1}-(-2)^{j+1}\right]}{5\cdot(-6)^j} \; \frac{5\cdot(-6)^{j+1}}{\left[(3)^{j+2}-(-2)^{j+2}\right]} \right| = \underset{j\to\infty}{\text{Lim}} \left| \frac{(-6)\left[(3)^{j+1}-(-2)^{j+1}\right]}{(3)^{j+2}-(-2)^{j+2}} \right|$$

Vasco suggests to let a = 3 and b = –2 in order to simplify the algebra. Then

$$R_Q = \underset{j\to\infty}{\text{Lim}} \left| \frac{-6\left[a^{j+1}-b^{j+1}\right]}{a^{j+2}-(-2)^{j+2}} \right| = \underset{j\to\infty}{\text{Lim}} \left| \frac{-6\,a^{j+1}\left(1-b^{j+1}/a^{j+1}\right)}{a^{j+2}\left(1-b^{j+2}/a^{j+2}\right)} \right|$$

Since $|b/a| < 1$, $\frac{b^n}{a^n}$ goes to zero as n goes to infinity, leaving

$$R_Q = \underset{j\to\infty}{\text{Lim}} \left| \frac{-6}{a} \right| = 2,$$

which agrees with the result of R_g. Now multiply P and Q for the purpose of using the product to calculate the radius of convergence.

$$P(z) = 2z + 3\sum_{j=2}^{\infty} z^j$$

$$= 2z + 3z^2 + 3z^3 + 3z^4 + 3z^5 + \cdots$$

$$Q(z) = \sum_{j=1}^{\infty} \frac{\left[(3)^{j+1}-(-2)^{j+1}\right]z^j}{5\cdot(-6)^j}$$

$$= -\frac{z}{6} + \frac{7z^2}{36} - \frac{13z^3}{216} + \frac{55z^4}{1296} - \frac{133z^5}{7776} + \frac{463z^6}{46656}$$

Multiply P(z) Q(z).

$$-2z\,\frac{z}{6} + 2z\,\frac{7z^2}{36} - 2z\,\frac{13z^3}{216} + 2z\,\frac{55z^4}{1296} - 2z\,\frac{133z^5}{7776} + \cdots$$

$$-3z^2\,\frac{z}{6} + 3z^2\,\frac{7z^2}{36} - 3z^2\,\frac{13z^3}{216} + 3z^2\,\frac{55z^4}{1296} - \cdots$$

$$-3z^3\,\frac{z}{6} + 3z^3\,\frac{7z^2}{36} - 3z^3\,\frac{13z^3}{216} + \cdots$$

$$-3z^4\,\frac{z}{6} + 3z^4\,\frac{7z^2}{36} - \cdots$$

$$-3z^5\,\frac{z}{6} + \cdots$$

$$P(z)Q(z) = -\frac{z^2}{3} + \left(\frac{14z^3}{36} - \frac{z^3}{2}\right) + \left(-\frac{26z^4}{216} + \frac{21z^4}{36} - \frac{z^4}{2}\right) + \left(\frac{110z^5}{1296} - \frac{39z^5}{216} + \frac{21z^5}{36} - \frac{z^5}{2}\right) + \ldots$$

$$= -\frac{z^2}{3} - \frac{z^3}{9} - \frac{z^4}{27} - \frac{z^5}{81} - \frac{z^6}{243} - \ldots.$$

The pattern of increasing powers of 3 by steps of 1 in the denominator is clear. We can write

$$P(z)\,Q(z) = \sum_{j=2}^{\infty} \frac{-z^j}{3^{j-1}}$$

$$R_{PQ} = \lim_{j\to\infty} \left| \frac{-\left(1/3^{j-1}\right)}{-\left(1/3^j\right)} \right| = \lim_{j\to\infty} \left| \frac{3^j}{3^{j-1}} \right| = 3$$

This is what we expected based on f(z) and g(z).