Needham, Chapter 2, Exercise 14, p. 114
G. Palmer, July 28, 2020, 6:45 pm PST

14 Consider the two power series, $P(z) = \sum_{j=0}^{\infty} p_j z^j$ and $Q(z) = \sum_{j=0}^{\infty} q_j z^j$, which have approximating polynomials $P_n(z) = \sum_{j=0}^{n} p_j z^j$ and $Q_m(z) = \sum_{j=0}^{m} q_j z^j$. If the radii of convergence of P(z) and Q(z) are R_1 and R_2 then both series are uniformly convergent in the disc $|z| \leq r$, where $r < \min\{R_1, R_2\}$. Thus if ϵ is the maximum error we will tolerate in this disc, we can find a sufficiently large n such that

$$P_n(z) = P(z) + \varepsilon_1(z) \text{ and } Q_n(z) = Q(z) + \varepsilon_2(z),$$

where the (complex) errors $\varepsilon_{1,2}(z)$ both have lengths less than ϵ. Use this to show that by taking a suffiently high value of n we can approximate $[P(z) + Q(z)]$ and $P(z)\,Q(z)$ with arbitrarily high precision using $[P_n(z) + Q_n(z)]$ and $P_n(z)\,Q_n(z)$, respectively.

Let $\varepsilon_1 = |P - P_n|$, and let $\varepsilon_2 = |Q - Q_m|$. P and Q are two unique infinite power series that converge to two corresponding complex functions, say f and g. Since they converge, as n or m becomes larger, the error (distance of P_n from P = f or Q_n from Q = g) becomes smaller, until in the limit as n or m goes to infinity, the error becomes zero. By using $[P_n + Q_n]$ *and* $P_n\,Q_n$ we ensure that the error of both approximations, P_n and Q_n shrinks as n goes to infinity. Since the total error $(\varepsilon_1 + \varepsilon_2)$ in the case of $[P_n + Q_n]$ or $|P\varepsilon_2 + Q\varepsilon_1 + \varepsilon_1\varepsilon_2|$ in the case of $P_n\,Q_n$ goes to 0 as n goes to infinity, it follows that there exists an n such that $(\varepsilon_1 + \varepsilon_2) < \epsilon$ and there exists an n such that $|P\varepsilon_2 + Q\varepsilon_1 + \varepsilon_1\varepsilon_2| < \epsilon$. It follows that we need only increase n until $\varepsilon_1 + \varepsilon_2 < \epsilon$ or $|P\varepsilon_2 + Q\varepsilon_1 + \varepsilon_1\varepsilon_2| < \epsilon$. This means we can arbitrarily decide on the value of ϵ and achieve virtual arbitrarily high precision. Actual precision is limited only by the precision of one's computer.

The case of $[P + Q]$ approximated with $[P_n + Q_n]$:

$$P_n + Q_n = P + \varepsilon_1 + Q + \varepsilon_2(z) = (P(z) + Q(z)) + (\varepsilon_1(z) + \varepsilon_2(z))$$

$$\lim_{n\to\infty} [\varepsilon_1 + \varepsilon_2] = \lim_{n\to\infty} |P_n - P| + \lim_{n\to\infty} |Q_n - Q| = 0 + 0 = 0$$

The case of PQ approximated with $P_n\,Q_n$:

$$P_n\,Q_n = (P + \varepsilon_1)\,(Q + \varepsilon_2) = PQ + P\varepsilon_2 + Q\varepsilon_1 + \varepsilon_1\varepsilon_2$$

This shows that the total error is $|P\varepsilon_2 + Q\varepsilon_1 + \varepsilon_1\varepsilon_2|$.

$$\lim_{n\to\infty} |P\varepsilon_2 + Q\varepsilon_1 + \varepsilon_1\varepsilon_2| = 0$$