Needham, Chapter 2, Exercise 13, p. 114
G. Palmer, July 28, 2020, 2:40 pm PST

13. We have seen that if we set $P_n(z) = z^n$*, then the representation of a complex function f(z) as an infinite series* $\sum_{j=0}^{n} c_n P_n(z)$ *(i.e., a power series) is unique. This is not true, however, if* $P_n(z)$ *is just any old set of polynomials. The following example is taken (and corrected) from Boas [1987, p. 33]. Defining*

$$P_0(z) = -1, \text{ and } P_n(z) = \frac{z^{n-1}}{(n-1)!} - \frac{z^n}{n!} \quad (n = 1, 2, 3, \ldots)$$

show that

$$-2\,P_0 - P_1 + P_3 + 2\,P_4 + 3\,P_5 + \ldots = e^z = P_1 + 2\,P_2 + 3\,P_3 + 4\,P_4 + \ldots$$

One must see that the LHS is an infinite series, which can be rewritten

$$\sum_{j=0}^{\infty} (j-2)\,P_j$$

The RHS can be rewritten

$$\sum_{j=1}^{\infty} j\,P_j$$

Therefore

$$\sum_{j=0}^{\infty} (j-2)\,P_j = \sum_{j=1}^{\infty} j\,P_j$$

Expand the LHS

$$-\sum_{j=0}^{\infty} 2\,P_j + \sum_{j=0}^{\infty} j\,P_j$$

$$2 - 2\sum_{j=1}^{\infty} P_j + \sum_{j=1}^{\infty} j\,P_j \qquad \text{(because } P_0(z) = -1\text{)}$$

Calculate

$$\sum_{j=1}^{\infty} P_j = \sum_{j=1}^{\infty} \left(\frac{z^{j-1}}{(j-1)!} - \frac{z^j}{j!}\right)$$

$$= \sum_{j=1}^{\infty} \frac{z^{j-1}}{(j-1)!} - \sum_{j=1}^{\infty} \frac{z^j}{j!}$$

$$= \sum_{k=0}^{\infty} \frac{z^k}{k!} - \sum_{j=1}^{\infty} \frac{z^j}{j!} \qquad \text{(Change of index to k does not change value of } \sum_{j=1}^{\infty} \frac{z^{j-1}}{(j-1)!}.\text{)}$$

$$= 1 + \sum_{j=1}^{\infty} \frac{z^j}{j!} - \sum_{j=1}^{\infty} \frac{z^j}{j!} \qquad \text{(Change k back to j.)}$$

$= 1 \qquad\qquad\qquad\qquad\qquad\qquad (1)$

Calculate

$\sum_{j=1}^{\infty} j\, P_j = \sum_{j=1}^{\infty} \left(\frac{j\, z^{j-1}}{(j-1)!} - \frac{\mathrm{j}z^j}{j!} \right)$

$= \sum_{k=0}^{\infty} \frac{(k+1)\, z^k}{k!} - \sum_{j=1}^{\infty} \frac{\mathrm{j}z^j}{j!}$ (by setting k = j-1 in first term on LHS; it does not change value of term)

$= 1 + \sum_{j=1}^{\infty} \left(\frac{(j+1)\, z^j}{j!} - \frac{\mathrm{j}z^j}{j!} \right)$ (changing k to j does not change value)

$= 1 + \sum_{j=1}^{\infty} \frac{z^j}{j!} = \sum_{j=0}^{\infty} \frac{z^j}{j!} = e^z$ (2), from p. 79)

From (1) and (2), the LHS becomes

$\sum_{j=1}^{\infty} P_j = 1$

$\sum_{j=1}^{\infty} j\, P_j = e^z$

$2 - 2\sum_{j=1}^{\infty} P_j + \sum_{j=1}^{\infty} j\, P_j = 2 - 2 + \sum_{j=1}^{\infty} j\, P_j$

$= \sum_{j=1}^{\infty} j\, P_j = e^z$

Calculate the RHS.

$P_1 + 2\, P_2 + 3\, P_3 + 4\, P_4 + \ldots = \sum_{j=1}^{\infty} j\, P_j = e^z$

Then $-2\, P_0 - P_1 + P_3 + 2\, P_4 + 3\, P_5 + \ldots = e^z = P_1 + 2\, P_2 + 3\, P_3 + 4\, P_4 + \ldots$.

This is what we wanted. If we examine and compare coefficients on the two sides, we have

		LHS						RHS		
C_0	C_1	C_2	C_3	C_4	...	C_0	C_1	C_2	C_3	C_4 ...
–2	–1	0	1	2	...	0	1	2	3	4 ...

Both sets of coefficients produce the function e^z in the infinite series. This shows that there is no unique set of coefficients for an infinite series based on the polynomial $P_n(z) = \frac{z^{n-1}}{(n-1)!} - \frac{z^n}{n!}$. Approximations based on the first few members of each power series will differ.