Needham, Chapter 2, Exercise 12, p. 113-114.
G. Palmer, July 19, 2020, 7 am PST

12 Consider the geometric series $P(z) = \sum_{j=0}^{\infty} z^j$, which converges to $1/(1-z)$ inside the unit disc. The approximating polynomials in this case are $P_m(z) = \sum_{j=0}^{m} z^j$.

(i) Show that the error $E_m(z) = \frac{|z|^{m+1}}{|1-z|}$.

See p. 70 for the basic framework. For P(z) centred at the origin with approximating sum $P_m(z)$ of sufficiently high degree and |z| < R, the radius of convergence, which is 1,

$$E_m(z) = |P(z) - P_m(z)|$$

$$E_m(z) = |P(z) - P_m(z)| = |1/(1-z) - \sum_{j=0}^{m} z^j \Big|$$

$$= |1/(1-z) - \left(1 + z + z^2 + \ldots + z^m\right) \Big|$$

$$= |1/(1-z) - \left(1 + z + z^2 + \ldots + z^m\right) (1-z) / (1-z) \Big|$$

$$= |1/(1-z) - \left[\left(1 + z + z^2 + \ldots + z^m\right) - z\left(1 + z + z^2 + \ldots z^m\right)\right] \Big/ (1-z) \Big|$$

$$= |1/(1-z) - \left[\left(1 + z + z^2 + \ldots + z^m\right) - \left(z + z^2 + \ldots + z^{m+1}\right) \Big/ (1-z) \Big|$$

$$= |1/(1-z) - \left[\left(1 - z^{m+1}\right) \Big/ (1-z) \Big|$$

$$= |z^{m+1}/(1-z)|$$

$$= |z|^{m+1} / |1-z|$$

The answer may be found more quickly by using the formula for a geometric series $P_m(z)$, which works for complex numbers where r < R.

$$E_m(z) = \Big| \sum_{j=0}^{\infty} z^j - \sum_{j=0}^{m} z^j \Big| = \Big| \frac{1}{1-z} - \frac{1-z^{m+1}}{1-z} \Big| = \frac{|z|^{m+1}}{|1-z|}$$

(ii) If z is any fixed point in the disc of convergence, what happens to the error as m tends to infinity?

Where |z| < 1, $\lim\limits_{z \to 1} |z|^{m+1} / |1-z| = \frac{0}{\text{const.}} = 0$, so E_m goes to zero.

(iii) If we fix m, what happens to the error as z approaches the boundary point z = 1?

$$\lim_{z \to 1} |z|^{m+1} \;/\; |1-z| = \frac{1}{0} = \text{undefined}$$

The error goes to infinity.

(iv) Suppose we want to approximate this series in the disk |z| ≤ 0.9, and further suppose that the maximum error we will tolerate is $\epsilon = 0.01$. Find the lowest degree polynomial $P_m(z)$ that approximates P(z) with the desired accuracy throughout the disc.

The equation for error shows that the answer depends on arg(z), which determines |1-z|.

Max |1-z| = 1- (-.9) = 1.9 when arg(z) is π.
Min |1-z| = 1 - .9 = .1 when arg(z) is 0.

Calculate the number of iterations with Max |1-z|:

$(0.9)^{m+1}/1.9 = 0.01$

$(0.9)^{m+1} = (1.9)\ 0.01 = .019$

(m+1) Log(0.9) = Log(.019)   taking Log of both sides

m = Log(.019)/Log(0.9) – 1 = 37.6167 – 1 ≈ 36.6

Calculate the number of iterations with Min |1-z|:

$(0.9)^{m+1}/.1 = 0.01$

$(0.9)^{m+1} = (.1)\ 0.01 = .001$

(m+1) Log(0.9) = Log(.001)

m = Log(.001)/Log(0.9) – 1 = 65.563 - 1 ≈ 64.6

We see that we need more iterations when |1-z| is at a minimum. **m** must be an integer, and accuracy increases with iterations, so to guarantee $\epsilon \leq 0.01$, we take largest of the two nearest integers, m = 65. The lowest degree polynomial $P_m(z)$ that approximates P(z) throughout the disc within the accuracy $\epsilon = 0.01$ is 65.